ASPG
American Scientific Publishing Group

INTERNATIONAL JOURNAL OF NEUTROSOPHIC SCIENCE

Volume 0, 2019

Editor in Chief: Broumi Said & Florentin Smarandache

ISSN: 2690-6805

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International Journal of Neutrosophic Science (IJNS)

Published and typeset in American Scientific Publishing Group (ASPG) is a USA academic publisher, established as LLC company on 2019 at New Orleans, Louisiana, USA. ASPG publishes online scholarly journals that are free of submission charges.

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New Orleans, USA

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Editorial Board

Aim and Scope

International Journal of Neutrosophic Science (IJNS) is a peer-review journal publishing high quality experimental and theoretical research in all areas of Neutrosophic and its Applications. IJNS is published quarterly. IJNS is devoted to the publication of peer-reviewed original research papers lying in the domain of neutrosophic sets and systems. Papers submitted for possible publication may concern with foundations, neutrosophic logic and mathematical structures in the neutrosophic setting. Besides providing emphasis on topics like artificial intelligence, pattern recognition, image processing, robotics, decision making, data analysis, data mining, applications of neutrosophic mathematical theories contributing to economics, finance, management, industries, electronics, and communications are promoted. Variants of neutrosophic sets including refined neutrosophic set (RNS). Articles evolving algorithms making computational work handy are welcome.

Topics of Interest

IJNS promotes research and reflects the most recent advances of neutrosophic Sciences in diverse disciplines, with emphasis on the following aspects, but certainly not limited to:

- Neutrosophic sets
- Neutrosophic algebra
- Neutrosophic topolog
- Neutrosophic graphs
- Neutrosophic probabilities
- Neutrosophic tools for decision making
- Neutrosophic theory for machine learning
- Neutrosophic statistics
- Neutrosophic numerical measures
- Classical neutrosophic numbers
- A neutrosophic hypothesis
- The neutrosophic level of significance
- The neutrosophic confidence interval
- The neutrosophic central limit theorem
- Neutrosophic theory in bioinformatics and medical analytics
- Neutrosophic tools for big data analytics
- Neutrosophic tools for deep learning
- Neutrosophic tools for data visualization
- Quadripartitioned single-valued neutrosophic sets
- Refined single-valued neutrosophic sets

- Applications of neutrosophic logic in image processing
- Neutrosophic logic for feature learning, classification, regression, and clustering
- Neutrosophic knowledge retrieval of medical images
- Neutrosophic set theory for large-scale image and multimedia processing
- Neutrosophic set theory for brain-machine interfaces and medical signal analysis
- Applications of neutrosophic theory in large-scale healthcare data
- Neutrosophic set-based multimodal sensor data
- Neutrosophic set-based array processing and analysis
- Wireless sensor networks Neutrosophic set-based Crowd-sourcing
- Neutrosophic set-based heterogeneous data mining
- Neutrosophic in Virtual Reality
- Neutrosophic and Plithogenic theories in Humanities and Social Sciences
- Neutrosophic and Plithogenic theories in decision making
- Neutrosophic in Astronomy and Space Sciences

Homomorphism and Isomorphism in strong neutrosophic graphs

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Abstract

The concept of strong neutrosophic graph is developed with example and some of their properties are investigated in this paper. Some definitions, homomorphism, isomorphism theorems and propositions in strong neutrosophic graphs are established. Basic operations(like union, intersection etc.,) and complement of strong neutrosophic graphs are also derived here.

Keywords: Strong neutrosophic graph, Complete neutrosophic graph, Complement of a neutrosophic graph, Cartesian product, Isomorphism.

1.Introduction

The concept of graph theory was first introduced by Euler in 1736[7]. It is an useful tool for solving problems in different mathematical areas such as algebra, operations research, optimization and computer science. Rosenfeld(1975) introduced the notion of fuzzy graph and several fuzzy analogs of theoretic concepts such as paths, cycles and connectedness[18]. Fuzzy models are very useful to reduce the difference between the traditional numerical models used in science and engineering and symbolic models used in expert system. Atanassov introduced the concept of intuitionistic fuzzy sets as a generalisation of fuzzy sets [2,3]. The fuzzy sets give the degree of membership of an element in a given set (and the non membership degree equals one minus the degree of membership), while intuitionistic fuzzy sets give both a degree of membership and a degree of non membership which are more or less independent from each other with the condition that the sum of these two degrees is not greater than 1. Also many researchers established and studied about fuzzy graphs and intuitionistic fuzzy graphs in [4,8,9,10,11, 13,14,15,16]. Neutrosophic set proposed by Smarandache [20] is a powerful tool for dealing incomplete inconsistance, imprecision, uncertain, false and indeterminate problems in the real world whenever the fuzzy and intuitionistic fuzzy approaches fail in such type of situations and it is the generalization of classical sets, fuzzy set, intuitionistic fuzzy set and interval valued fuzzy set etc,. So Smarandache defined four main categories of neutrosophic graphs in [21, 22] to handle these type of situations. M.Mullai[23introduced the concept of domination

DOI: 10.5281/zenodo.3908341

Received: January 14, 2019 Revised: March 27, 2019 Accepted: April 28, 2019

in neutrosphic graphsIn this paper, strong neutrosophic graph is developed with examples. Also, some definitions, theorems and propositions related to opreations, homomorphism, isomorphism and complement of strong neutrosophic graphs are established.

2 Preliminaries

Basic definitions which are necessary to this article are reviewed in this section.

Definition 0.1 [5] A fuzzy graph $G:(\sigma,\mu)$ is a pair of functions $\sigma: S \to [0,1]$ and $\mu: S \times S \to [0,1]$, we have $\mu(x,y) \leq \sigma(x) \wedge \sigma(y)$ for all, $x, y \in S$, where $\wedge$ stands for minimum and σ is a subset of a nonempty set S and μ is a fuzzy relation on σ .

Definition 0.2 [12] A fuzzy graph G is connected if $\mu^{\infty}(u,v) > 0$ for all , $u, v \in S$.

Definition 0.3 [12] A fuzzy graph G is said to be a strong fuzzy graph if $\mu(x,y) = \sigma(x) \wedge \sigma(y)$, for all $(x,y) \in S \times S$.

Definition 0.4 [12] A fuzzy graph G is said to be a complete fuzzy graph if $\mu(x,y) = \sigma(x) \wedge \sigma(y)$, for all (x,y) ∈ S.

Definition 0.5 [6] An arc (x,y) is said to be a strong arc if $\mu(x,y) \geq \mu^{\infty}(x,y)$. A node X is said to be an isolated node if $\mu(x,y) = 0$, for all, $y \neq x$.

Definition 0.6 [5] A fuzzy subset μ on a set X is a map $\mu: X \to [0,1]$. A map $\nu: X \times X \to [0,1]$ is called a fuzzy relation on X if $\nu(x,y) \leq \min(\mu(x),\mu(y))$ for all, $x, y \in X$. A fuzzy relation $\nu(x,y) = \nu(y,x)$ for all, $x, y \in X$.

Definition 0.7 [5] Let $G:(\sigma,\mu)$ be a fuzzy graph.The complement of G is defined as $\overline{G}:(\sigma,\overline{\mu})$ where $\overline{\mu}(x,y) = \sigma(x) \wedge \sigma(y) - \mu(x,y)$, for all $x, y \in S$. When G is a fuzzy graph, $\overline{G}:(\sigma,\overline{\mu})$ is also a fuzzy graph.

Definition 0.8 [17] A mappings $A = (\mu_A, \nu_A): X \to [0,1] \times [0,1]$ is called an intuitionistic fuzzy set in X, if $\mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$, where the mapping $\mu_A: X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non membership (namely $\nu_A(x)$) of each element $x \in X$ to A , respectively.

Definition 0.9 [17] For every two intuitionistic fuzzy sets $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ in X , we define

$$(A \cap B)(x) = (\min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x)))).$$

$$(A \cup B)(x) = (\max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x)))).$$

Definition 0.10 [17] Let X be a non empty set. Then we call a mapping $A = (\mu_A, \nu_A): X \to [0,1] \times [0,1]$ an intuitionistic fuzzy relation on X if $A = (\mu_A(x,y) + \nu_A(x,y) \leq 1$ for all $(x,y) \in X \times X$.

Definition 0.11 [5] Let $A = (\mu_A, \nu_A)$ and $B = (\mu_A, \nu_B)$ be intuitionistic fuzzy sets on a set X. If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy relation on a set X, then $A = (\mu_A, \nu_A)$ is called an intuitionistic fuzzy relation on $B = (\mu_B, \nu_B)$ if

DOI: 10.5281/zenodo.3908341

$\mu_A(x,y) \leq \min(\mu_B(x), \mu_B(y))$ and $\nu_A(x,y) \geq \max(\nu_B(x), \nu_B(y))$ for all, $x, y \in X$. An intuitionistic fuzzy relation A on X is called symmetric if $\mu_A(x,y) = \mu_A(y,x)$ and $\nu_A(x,y) = \nu_A(y,x)$ for all, $x, y \in X$.

Definition 0.12 [19] Let X be a space of points (objects) with generic elements in X denoted by X , then the neutrosophic sets A (NS A) is an object having the form

$$A = \{< x: T_A(x), I_A(x), F_A(x) > 0\},\ x \in X,$$

where the functions $T, I, F: X \to]0^-, 1^+[$ define respectively the truth membership function, an indeterminacy membership function , and a falsity membership function of the element $x \in X$ the set A with condition

$$0^- \leq T_A(x) + I_A(x) + F_A(x) \leq 3 +$$

The function $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or non standard subsets of $]0^-, 1[= 0$.

Definition 0.13 [19] Let X be a space of points (objects) with generic elements in X is denoted by X. A single valued neutrosophic set A (SVNS A) is characterized by truth membership function $T_A(x)$, an indeterminacy membership function $I_A(x)$ and a falsity membership function $F_A(x)$. For each point x in X $T_A(x), F_A(x)$ and $I_A(x) \in [0,1]$. A SVNS A can be written as

$$A = \{< T_A(x), I_A(x), F_A x >, x \in X\}$$

Definition 0.14 [19] Let $A = (T_A, I_A, F_A)$ and $B = (T_B, I_B, F_B)$ be single valued neutrosophic sets on a set x. If $A = (T_A, I_A, F_A)$ is a single valued neutrosophic relation on a set X, then $A = (T_A, I_A, F_A)$ is called a single valued neutrosophic relation on $B = (T_B, I_B, F_B)$ if

$$T_B(x,y) \leq \min(T_A(x), T_A(y))$$

$$I_B(x,y) \geq \max(I_A(x), I_A(y))$$

$$\text{and } F_B(x,y) \geq \max\big(F_A(x), F_A(y)\big) \text{ for all, } x, y \text{ in X,}$$

A single valued neutrosophic relation A on X is called symmetric if

$$T_A(x,y) = T_A(y,x), I_A(x,y) = I_A(y,x)$$

$$F_A(x,y) = F_A(y,x), \text{and}$$

$$T_B(x,y) = T_B(y,x), I_B(x,y) = I_B(y,x)$$

$$F_B(x,y) = F_B(y,x) \text{for all, } x, y \in X,$$

Definition 0.15 [19] A single valued neutrosophic graph (SVN - graph) with underlying set V is defined to be a pair $G = (A, B)$ where,

DOI: 10.5281/zenodo.3908341

(i) The functions $T_A: V \to [0,1], I_A: \to [0,1] and F_A: V \to [0,1]$ denote the degree of truth membership, degree of indeterminacy membership, and degree of falsity membership of the element $v_i \in V$, respectively and

$$0 \leq T_A(v_i) + I_A(v_i) + F_A(v_i) \leq 3$$

for all, $v_i \in V(i = 1,2, \ldots . n)$ ii) The functions $T_B: E \subseteq V \times V \to [0,1], I_B: E \subseteq V \times V \to [0,1]$, and

$F_B: E \subseteq V \times V \to [0,1]$ are defined by

$$T_B(v_i, v_j) \leq \min(T_A(v_i), T_A(v_j)),$$

$$I_B(v_i, v_j) \geq \max(I_A(v_i), I_A(v_j), \text{and}$$

$$F_B(v_i, v_j) \geq \max(F_A(v_i), F_A(v_j).$$

denote the degree of truth membership , degree of indeterminacy membership ,and degree of falsity membership of the edge $(v_i), v_j) \in E$ respectively , where

$$0 \leq T_B(v_i, v_j) + I_B(v_i, v_j) + F_B(v_i, v_j) \leq 3,$$

for all, $(v_i), (v_j) \in E(i, j = 1,2, .. n)$.

Definition 0.16 [3] We say a fuzzy subgraph $H = (\sigma, \tau)$ is a full spanning fuzzy subgraph $G = (\sigma, \mu)$ on (V,X) if H is a spanning fuzzy subgraph of G and for all, $u, v \in V$ either $\tau(u, v) = 0$ or $\tau(u, v) = \mu(u, v)$.

3 Strong neutrosophic graphs

The concept of strong neutrosophic graph with examples, operations, properties, propositions, homomorphism and isomorphism theorems on strong neutrosophic graphs are introduced and discussed briefly here.

Definition 0.17 [19] An neutrosophic graph G = (A,B) is called strong neutrosophic graph if

$$T_B(xy) = \min(T_A(x), T_A(y)) and I_B(xy) = \max(I_A(x), I_A(y)), \text{and}$$

$$F_B(xy) = \max(F_A(x), F_A(y)) \text{ for all, } xy \in E.$$

Definition 0.18 [19]

Let $A = (T_A, I_A, F_A)$ and $A' = (T'_A, I'_A, F'_A)$ be neutrosophic fuzzy subsets of V_1 and V_2 and let $B = (T_B, I_B, F_B)$ and $B' = (T'_B, I'_B, F'_B)$ neutrosophic subsets of E_1 and E_2, respectively. The cartesian product of two strong neutrosophic graphs G_1^* and G_2^* is denoted by $G_1 \times G_2 = (A \times A', B \times B')$ and is defined as follows :

$$1. \begin{cases} ((T_A \times T'_A)(x_1, x_2) = \min((T_A(x_1, T'_A(x_2)) \\ ((I_A \times I'_A)(x_1, x_2) = \max((I_A(x_1, I'_A(x_2)) \quad \text{forall}, (x_1, x_2) \in V , \\ ((F_A \times F'_A)(x_1, x_2) = \max((F_A(x_1, F'_A(x_2)), \end{cases}$$

DOI: 10.5281/zenodo.3908341

2. $\begin{cases} ((T_B \times T'_B)((x, x_2), (x, y_2)) = \min(T_A(x), T'_B(x_2, y_2)) \\ ((I_B \times I'_B)((x, x_2), (x, y_2)) = \max(I_A(x), I'_B(x_2, y_2)) \\ ((F_B \times F'_B)((x, x_2), (x, y_2)) = \max(F_A(x), F'_B(x_2, y_2)), \end{cases}$

for all, $x \in V_1 (x_1, x_2) \in E_2$

3. $\begin{cases} ((T_B \times T'_B)(x_1, z), (y_1, z)) = \min((T_B(x_1, y_1), T'_B(z)) \\ ((I_B \times I'_B)(x_1, z), (y_1, z)) = \max((I_B(x_1, y_1), I'_B(z)) \\ ((F_B \times I'_B)(x_1, z), (y_1, z)) = \max((F_B(x_1, y_1), F'_B(z)), \end{cases}$

for all, $z \in V_2$ for all, $x_1 y_2 \in E_1$.

Proposition 0.19 If G_1 and G_2 are the strong neurotrosophic graphs, then $G_1 \times G_2$ is a strong neutrosophic graph.

Proof:

It is a straight forward.

Proposition 0.20 If $G_1 \times G_2$ is strong neutrosophic graph, then at least G_1 or G_2 must be strong .

Proof:

Suppose that G_1 and G_2 are not strong neutrosophic graphs.

Then there exit $x_1 y_1 \in E_1$ and $x_2 y_2 \in E_2$ such that,

$$(T_{B_1}(x_1 y_1)) < min(T_{A_1}(x), T_{A_1}(y)), (T_{B_2}(x_1 y_1) < min(T_{A_2}(x), T_{A_2}(y))$$

$(I_{B_1}(x_1 y_1) > max(I_{A_1}(x), I_{A_1}(y)), (I_{B_2}(x_1 y_1) > max(I_{A_2}(x), I_{A_2}(y))$

$(F_{B_1}(x_1 y_1) > max(F_{A_1}(x), F_{A_1}(y)), (F_{B_2}(x_1 y_1)) > max(F_{A_2}(x), F_{A_2}(y))$

Assume that, $T_{B_2}(x_2 y_2) \leq T_{B_1}(x_1, y_1) < min(T_{A_1}(x_1), T_{A_1}(y_1)) \leq T_{A_1}(x_1)$ Let $E = (x, x_2), (x, y_2)/x_1 \in V_1, (x_2 y_2) \in E_2 \cup ((x_1, z), (y_1, z))/z \in V_2, \forall, x_1 y_1 \in E_1$.

Consider, $(x, x_2), (x, y_2) \in E$,

We have, $(T_{B_1} \times T_{B_2}((x, x_2), (x, y_2)) = \min(T_{A_1}(x), T_{B_2}(x_2 y_2))$

$< (T_{A_1}(x), T_{A_2}(x_2), T_{A_2}(y_2))$and

$(T_{A_1} \times T_{A_2})((x_1, x_2)) = \min(T_{A_1}(x_1), T_{A_2}(x_2), T_{A_1} \times T_{A_2})(x_1, y_2)$

$= \min(T_{A_1}(x_1), T_{A_2}(y_2))$

Therefore,

$\min(T_{A_1} \times T_{A_2})(x, x_2), T_{A_1} \times T_{A_2}(x, y_2) = \min(T_{A_1}(x), T_{A_2})(x_2), T_{A_2}(y_2))$

Hence $(T_{B_1} \times T_{B_2})((x, x_2), (x, y_2)) < min((T_{A_1} \times T_{A_2})(x, x_2), (T_{A_1} \times T_{A_2})(x, y_2))$

Similarly , we can easily show that

DOI: 10.5281/zenodo.3908341

$$(I_{B_1} \times I_{B_2})((x, x_2), (x, y_2)) > max(I_{A_1} \times I_{A_2}(x, x_2)), (I_{A_1} \times I_{A_2})(x, y_2))$$

$$(F_{B_1} \times F_{B_2})((x, x_2), (x, y_2)) > max(F_{A_1} \times F_{A_2})(x, x_2), (F_{A_1} \times F_{A_2})(x, y_2))$$

That is $G_1 \times G_2$ is not strong neutrosophic graph, a contradiction . Hence, this ends the proof .

Remark 0.21 If G_1 is strong and G_2 is not strong , $G_1 \times G_2$ may or may not be strong.

Proposition 0.22 Let G_1 be a strong neutrosophic graph of G_1^* . Then for every neutrosophic graph G_2 of G_2^* , $G_1 \times G_2$ is strong if and only if

$T_A(x_1) \leq T_B(x_2y_2)$, $I_A(x_1) \leq I_B(x_2y_2)$ and

$F_A(x_1) \leq F_B(x_2y_2)$, for all, $x_1 \in V_1$ and $x_2y_2 \in E_2$.

Definition 0.23 [**19**]

Let $A = (T_A, I_A, F_A)$ and $A' = (T'_A, I'_A, F'_A)$ be neutrosophic subsets of V_1 and V_2 and let $B = (T_B, I_B, F_A)$and$B' = (T'_B, I'_B, F'_A)$ be neutrosophic subsets of E_1 and E_2 , respectively. The composition of two strong neutrosophic graphs G_1 and G_2 of the graphs G_1^* and G_2^* is denoted by $G_1[G_2] = (AoA', BoB')$ and is defined as follows :

1. $$\begin{cases} (T_A o T'_A)(x_1, x_2) = \min(T_A(x_1, T'_A(x_2)) \\ (I_A o I'_A)(x_1, x_2) = \max((I_A(x_1, I'_A(x_2)) \\ (F_A o F'_A)(x_1, x_2) = \max((F_A(x_1, F'_A(x_2)) \end{cases}$$

for all, $(x_1, x_2) \in V$,

2. $$\begin{cases} (T_B o T'_B)(x, x_2), (x, y_2) = \min(T_A(x), T'_B(x_2y_2)) \\ (I_B o I'_B)(x, x_2), (x, y_2) = \max((I_A(x), I'_B(x_2y_2)) \\ (F_B o F'_B)(x, x_2), (x, y_2) = \max(F_A(x), F'_B(x_2y_2)) \end{cases}$$

for all, $x \in V_1, \text{forall}, x_2y_2 \in E_2$

3. $$\begin{cases} (T_B o T'_B)(x_1, z), (y_1, z) = \min(T_B(x_1y_1), T'_A(z) \\ (I_B \times I'_B)(x_1, z), (y_1, z) = \max(I_B(x_1y_1), I'_A(z)) \\ (F_B \times F'_B)(x_1, z), (y_1, z) = \max(F_B(x_1y_1), F'_A(z)) \end{cases}$$

for all, $z \in V_2$, for all $x_1y_1 \in E_1$.

4. $$\begin{cases} (T_B o T'_B)((x_1, x_2), (y_1, y_2)) = \min(T'_A(x_2)T'_A(y_2), T_B(x_1y_1)), \\ ((I_B o I'_B)((x_1, x_2)(y_1, y_2)) = \max(I'_A(x_2), I'_A(y_2), I_B(x_1y_1)), \\ ((F_B o F'_B)((x_1, x_2)(y_1, y_2)) = \max(F'_A(x_2), F'_A(y_2), F_B(x_1y_1)), \end{cases}$$

for all, $((x_1, x_2)(y_1, y_2) \in E^0 - E$

We state the following propositions without proofs .

Proposition 0.24 If G_1 and G_2 are the strong neutrosophic graphs , then $G_1[G_2]$ is a strong neutrosophic graph.

Proposition 0.25$G_1[G_2]$ is strong neutrosopsic graph , then at least G_1 or G_2 must be strong.

DOI: 10.5281/zenodo.3908341

Definition 0.26 [19]$A = (T_A, I_A, F_A)$ and $A' = (T'_A, I'_A, F'_A)$ be neutrosophic fuzzy subsets of V_1 and V_2 and let $B = (T_B, I_B, F_A)$and$B' = (T'_B, I'_B, F'_B)$ be neutrosophic subsets of E_1 and E_2 , respectively. The joint of two strong neutrosophic graphs G_1 and G_2 of the graphs G_1^* and G_2^* is denoted by $G_1[G_2] = (A + A', B + B')$ and is defined as follows :

1. $\begin{cases}(T_A + T'_A)(x) = (T_A + T'_A)(x)),\\(I_A + I'_A)(x) = (I_A + I'_A)(x)),\\(I_A + I'_A)(x) = (I_A + I'_A)(x)),\end{cases}$

if $x \in V_1 \cup V_2$,

2. $\begin{cases}(T_B + T'_B)(xy) = (T_B \cup T'_B)(xy) = T_B(xy),\\(I_B + I'_B)(xy) = (I_B \cap I'_B)(xy) = I_B(xy),\\(F_B + F'_B)(xy) = (F_B \cap F'_B)(xy) = F_B(xy),\end{cases}$

if $xy \in E_1 \cup E_2$,

3. $\begin{cases}(T_B + T'_B)(xy) = \min(T_A(x), T'_A)(y),\\(I_B + I'_B)(xy) = \max(I_A(x), I'_A)(y),\\(F_B + F'_B)(xy) = \max(F_A(x), F'_A)(y),\end{cases}$

if $xy \in E'$.

Proposition 0.27 If G_1 and G_2 are the strong neutrosophic graphs, then $G_1 + G_2$ is a strong neutrosophic graph.

Definition 0.28 [19]$A = (T_A, I_A, F_A)$ and $A' = (T'_A, I'_A, F'_A)$ be neutrosophic subsets of v_1 and V_2 and let $B = (T_B, I_B, F_A)$ and $B' = (T'_B, I'_B, F'_A)$ be neutrosophic subsets of E_1 and E_2 , respectively . The union of two strong neutrosophic graphs $G_1 \cup G_2$ of the graphs G_1^* and G_2^* is denoted by $G_1 \cup G_2 = (A \cup A', B \cup B')$ and is defined as follows :

1. $\{(T_A \cup T'_A)(x) = T_A(x), x \in V_1 \cup \overline{V}_2, (T_A \cup T'_A)(x) = T'_A(x), x \in V_2 \cup \overline{V}_2 (T_A \cup T'_A)(x) = \max(T_A(x), T'_A(x)), x \in V_1 \cup V_2,$

2. $\{(I_A \cap I'_A)(x) = I_A(x), x \in V_1 \cap \overline{V}_2, (I_A \cap I'_A)(x) = I'_A(x), x \in V_2 \cap \overline{V}_2, (I_A \cap I'_A)(x) = \min(I_A(x), I'_A(x)), x \in V_1 \cap V_2,$

3. $\{(F_A \cap F'_A)(x) = F_A(x), x \in V_1 \cap \overline{v}_2, (F_A \cap F'_A)(x) = F'_A(x), x \in V_2 \cap \overline{V}_2, (F_A \cap F'_A)(x) = \min(F_A(x), F'_A(x)), x \in V_1 \cap V_2,$

4. $\{(T_A \cup T'_A)(xy) = T_B(xy), xy \in E_1 \cap \overline{E}_2, (T_A \cup T'_A)(xy) = T'_B(xy), xy \in E_2 \cap \overline{E}_1, (T_A \cup T'_A)(xy) = \max(T_B(x), T'_B(x)), x \in E_1 \cap E_2,$

5. $\{(I_B \cap I'_B)(xy) = I_B(xy), x \in E_1 \cap \overline{E}_2 (I_B \cap I'_B)(xy) = I'_B(xy), x \in E_2 \cap \overline{E}_1 (I_B \cap I'_B)(xy) = \max(I_B(xy), I'_B(xy)), xy \in E_1 \cup E_2,$

6. $\{(F_B \cap F'_B)(xy) = F_B(xy), x \in E_1 \cap \overline{E}_2, (F_B \cap F'_B)(xy) = F'_B(xy), x \in E_2 \cap \overline{E}_1, (F_B \cap F'_B)(xy) = \max(F_B(xy), F'_B(xy)), xy \in E_1 \cap E_2,$

DOI: 10.5281/zenodo.3908341

Definition 0.29 [19] The complement of a strong neutrosophic graph $G = (A, B)$ of $G^* = (V, E)$ is a strong neutrosophic graph $\overline{G} = (\overline{A}, \overline{B})$ on G^* , where $\overline{A} = (\overline{T}_A, \overline{I}_A, \overline{F}_B)$ and $\overline{B} = (\overline{T}_B, \overline{I}_B, \overline{F}_B)$ are defined by

(i) $\overline{V} = V$,

(ii) $\overline{T}_A(x) = T_A(x), \overline{I}_A(x) = I_A(x) \overline{F}_A(x) = F_A(x)$ for all, $x \in V$, (iii)

1. $\overline{T}_A(xy) = \begin{cases} 0 & \text{if} T_B(xy) > 0 \\ \min(T_A(x), T_A(y) & \text{if} T_B(xy) = 0 \end{cases}$

2. $\overline{I}_A(xy) = \begin{cases} 0 & \text{if} I_B(xy) > 0 \\ \max(I_A(x), I_A(y) & \text{if} I_B(xy) = 0 \end{cases}$

3. $\overline{F}_A(xy) = \begin{cases} 0 & \text{if} F_B(xy) > 0 \\ \max(F_A(x), F_A(y) & \text{if} F_B(xy) = 0 \end{cases}$

Remark 0.30 If $G = (A, B)$ is an neutrosophic graph of $G^* = (V, E)$ Then from definition 12 , it follows that

$\overline{\overline{G}}$ is given by the neutrosophic graph $\overline{\overline{G}} = (\overline{\overline{A}}, \overline{\overline{B}})$ on $G^* = (V, E)$

where $\overline{\overline{A}} = A$ and

$$\overline{\overline{T}}_B(xy) = \min(T_A(x), T_A(y)) \overline{\overline{I}}_B(xy) = \max(I_A(x), I_A(y)) \text{and}$$

$$\overline{\overline{F}}_B(xy) = \max(F_A(x), F_A(y)), \text{for all, } xy \in E.$$

Thus $\overline{\overline{T}}_B = T_B$ and $\overline{\overline{I}}_B = I_B, \overline{\overline{F}}_B = F_B$ on V where $B = (T_B, I_B, F_B)$ is the strongest neutrosophic graph relation on A.

For any neutrosophic graph $G, \overline{\overline{G}}$ is strong neutrosophic graph and $G \subseteq \overline{\overline{G}}$.

The following propositions are obvious.

Proposition 0.31 $G = \overline{\overline{G}}$ if and only if G is a strong neutrosophic graph .

Proposition 0.32 Let $G = (A_i, B_i)$ be a strong neutrosophic graph of $G_i^* = (V_i, E_i)$ for i = 1,2.

Then the following are true :

(a) $G_i \subseteq \overline{\overline{G}}_i$.

(b) $\overline{G}_i = (\overline{\overline{G}}_i)$,

(c) If $G_1 \subseteq G_2$, then $\overline{\overline{G}}_1 \subseteq \overline{\overline{G}}_2$.

$\overline{\overline{G}}$ is the smaller strong neutrosophic graph that contains $G_i^* = (V, E)$.

Definition 0.33 [23] A strong neutrosophic graph G is called self complementary if $G \approx \overline{G}$.

Proposition 0.34 Let G be a self complementary strong neutrosophic graph. Then

DOI: 10.5281/zenodo.3908341

$$\sum_{x\neq y} T_B(xy) = \sum_{x\neq y} \min(T_A(x), T_A(y))$$

$\sum_{x\neq y} I_B(xy) = \sum_{x\neq y} \max(I_A(x), I_A(y))$.

$\sum_{x\neq y} F_B(xy) = \sum_{x\neq y} \max(F_A(x), F_A(y))$.

Proposition 0.35 Let G be a strong neutrosophic graph. $T_B(xy) = \min(T_A(x), T_A(y))$, for all, $x, y \in V$, then G is self complementary .

Proof:

Let G be a strong neutrosophic graph such that

$$T_B(xy) = \min(T_A(x), T_A(y))$$

$$I_B(xy) = \max(I_A(x), I_A(y))$$

$$F_B(xy) = \max(F_A(x) \text{for all}, \ x, y \in V.$$

Then $G \approx \overline{G}$ under the identity map $I: V \to V$.

Hence ,G is self complementary .

Proposition 0.36 Let G_1 and G_2 be strong neutrosophic graphs. Then $G_1 \approx G_2$ if and only if

$\overline{G}_1 \approx \overline{G}_2$.

Proof:

$T_{A_1}(x) = T_{A_2}(f(x)), I_{A_1}(x) = I_{A_2}(f(x))andF_{A_1}(x) = F_{A_2}(f(x))$ for all, $x \in V_1$,

$T_{B_1}(xy) = T_{B_2}(f(x)f(y)), I_{B_1}(xy) = I_{B_2}(f(x)f(y))$, and

$F_{B_1}(xy) = F_{B_2}(f(x)f(y))$, for all, $xy \in E_1$.

By definition of complement , we have

$\overline{T}_{B_1}(xy) = \min(T_{A_1}(x), T_{A_1}(y)) = \min(T_{A_2}(f(x)), T_{A_2}(f(y))) = \overline{T}_{B_2}(f(x)f(y))$,

$\overline{I}_{B_1}(xy) = \max(I_{A_1}(x), I_{A_1}(y)) = \max(I_{A_2}(f(x)), I_{A_2}(f(y))) = \overline{I}_{B_2}(f(x)f(y))$,

$\overline{F}_{B_1}(xy) = \max(F_{A_1}(x), F_{A_1}(y)) = \max(F_{A_2}(f(x)), F_{A_2}(f(y))) = \overline{F}_{B_2}(f(x)f(y))$,

For all, $xy \in E_1$.

Hence, $\overline{G}_1 \approx \overline{G}_2$

DOI: 10.5281/zenodo.3908341

The proof of the converse part is straight forward. This completes the proof.

Definition 0.37 [23] An neutrosophic fuzzy graph G = (A,B) is called complete if

$T_B(xy) = \min(T_A(x), T_A(y))$ and , $I_B(xy) = \max(I_A(x), I_A(y))$

$F_B(xy) = \max(F_A(x), F_A(y))$ for all, $xy \in E$.

We use the notion $C_m(G)$ for a complete neutrosophic fuzzy graph where $|V| = m$.

Proposition 0.38 An neutrosophic graph $G = (A, B)$ is called bigraph if and only if there exists neutrosophic graphs $G_i = (A_i, B_i)$ for $i = 1,2$ of

G = (A,B) such that $G = (A, B)$ is the join $G_1 + G_2$

where $V_1 \cap V_2 = \emptyset$ and $E_1 \cap E_2 = \emptyset$.

An neutrosophic bigraph is said to be complete if and only if

$$T_B(xy) > 0, I_B(xy) > 0, F_B(xy) > 0 \forall xy \in E$$

We use the notion $C_{m,n}$ (G) for a bigraph, where $|V_1| = m$ and $|V_2| = n$.

Proposition 0.39 $C_{m,n}(G) = C_m(G_1) + C_n(G_2)$.

Proof:

It is a straight forward.

Definition 0.40 [19] Let G_1 and G_2 be the strong neutrosophic graphs . A homomorphism $f: V_1 \to V_2$ which satisfies the following conditions :

(a) $T_{A_1}(x_1) = T_{A_2}(f(x_1)), I_{A_1}(x_1) = I_{A_2}(f(x_1))$, and

$F_{A_1}(x_1) = F_{A_2}(f(x_1))$,

(b) $T_{B_1}(x_1y_1) = T_{B_2}(f(x_1)f(y_1), I_{B_1}(x_1y_1) = I_{B_2}(f(x_1)f(y_1)$,and

$F_{B_1}(x_1y_1) = F_{B_2}(f(x_1)f(y_1)$, for all $x_1 \in V_1, x_1y_1 \in E_1$

Definition 0.41 [19] Let G_1 and G_2 be strong neutrosophic graphs . isomorphism $f: G_1 \to G_2$ is bijective mapping $f: V_1 \to V_2$ which satisfies the following conditions :

(c) $T_{A_1}(x_1) = T_{A_2}(f(x_1)), I_{A_1}(x_1) = I_{A_2}(f(x_1))$, and

DOI: 10.5281/zenodo.3908341

$F_{A_1}(x_1) = F_{A_2}(f(x_1))$,

(d)$T_{B_1}(x_1y_1) = T_{B_2}(f(x_1)f(y_1), I_{B_1}(x_1y_1) = I_{B_2}(f(x_1)f(y_1)$ and

$F_{B_1}(x_1y_1) = F_{B_2}(f(x_1)f(y_1)$, for all, $x_1 \in V_1, x_1, y_1 \in E_1$.

Definition 0.42 [**19**] Let G_1 and G_2 be strong neutrosophic graphs . Then a weak isomorphism $f: G_1 \to G_2$ is bijective mapping $f: V_1 \to V_2$ which satisfies the following conditions :

(e) f is homomorphism ,

(f) $T_{A_1}(x_1) = T_{A_2}(f(x_1)), I_{A_1}(x_1) = I_{A_2}(f(x_1)), \text{and} F_{A_1}(x_1) = F_{A_2}(f(x_1))$, for all, $x_1 \in V_1$.

Thus , a weak isomorphism preserves the weights of the nodes but not necessarily the weights of the arcs.

Definition 0.43 [**19**] Let G_1 and G_2 be strong neutrosophic graphs . A co weak isomorphism $f: G_1 \to G_2$ is bijective mapping $f: V_1 \to V_2$ which satisfies the following conditions :

(g) f is homomorphism ,

(h)$T_{B_1}(x_1y_1) = T_{B_2}(f(x_1)f(Y_1)), I_{B_1}(x_1y_1) = I_{B_2}(f(x_1)f(y_1))$, and

$F_{B_1}(x_1y_1) = F_{B_2}(f(x_1)f(y_1))$, for all $x_1 \in V_1$.

Thus, a co weak isomorphism preserves the weights of the arcs but not necessarily the weights of the nodes.

Remark 0.44

1. If $G_1 = G_2 = G$, then the homomorphism f over itself is called an endomorphism . An isomorphism f over G is called an automorphism
2. Let $A = (T_A, I_A, F_A)$ be a strong neutrosophic graph with an underlying set V . Let Aut(G) be the set of all strong neutrosophic automorphism of G . Let $e: G \to G$ be a map defined by $e(x) = x$ for all $x \in V$.

   Clearly , $e \in \text{Aut}(G)$.
3. $G_1 = G_2$, then the weak and co weak isomorphism actually become isomorphic.
4. If $V_1 \to V_2$ is a bijective map , then $f^-: V_1 \to V_2$ is also a bijective map.

We state the following propositions without their proofs.

Proposition 0.45 Let G_1 and G_2 be strong neutrosophic graphs. If there is a weak isomorphism between G_1 and G_2 , then there is a weak isomorphism between $\overline{G}_1$ and $\overline{G}_2$.

DOI: 10.5281/zenodo.3908341

Proposition 0.46 Let G_1 and G_2 be strong neutrosophic graphs. If there is a weak isomorphism between G_1andG_2 , then there is a weak isomorphism between $\overline{G}_1$ and $\overline{G}_2$.

4 Conclusion

A neutrosophic set is a generalization of the notion of a fuzzy set.neutrosophic models give more precision,flexibility and compatibility to the system as compared to the classic and fuzzy models.We have introduced the concepts of (i) strong neutrosophic graphs ,and have presented some of their properties in this paper.It is clear that the most of these results can be simply extented to (S,Y)-fuzzy graphs , whereS and T are given imaginable triangular norms.The obtained results can be applied in various areas of engineering,computer science : artificial intelligence,signal processing,pattern recognition,robotics, computer networks,expert systems,and medical diagnosis.Our future plan to extend our research of fuzzification to (1) Bipolar fuzzy hypergraphs;(2) neutrosophic hypergraphs;(3) Vague hypergraphs ; (4) Interval-valued hypergraphs;(5) Soft fuzzy hypergraphs.

Acknowledgements

The article has been written with the joint financial support of RUSA-Phase 2.0 grant sanctioned vide letter No.F 24-51/2014-U, Policy (TN Multi-Gen), Dept. of Edn. Govt. of India, Dt. 09.10.2018, UGC-SAP (DRS-I) vide letter No.F.510/8/DRS-I/2016(SAP-I) Dt. 23.08.2016 and DST (FST - level I) 657876570 vide letter No.SR/FIST/MS-I/2018-17 Dt. 20.12.2018.

REFERENCES

[1] A.Alaoui, On fuzzification of some concepts of graphs , Fuzzy Sets and Systems 101,363-389, 1999.

[2] K.T.Atanassov, Intuitionistic fuzzy sets : Theory,applications,Studies in fuzziness and soft computing, Heidelberg,Newyork,physica-verl.,1999.

[3] K.T.Atanassov, Intuitionistic fuzzy sets,Fuzzy Sets and Systems, 20,1986.

[4] P.Bhattacharya, Some remarks on fuzzy graphs, Pattern Recognition Letters, 6, pp.297-302,1987.

[5] K.R. Bhuttani, On automorphisms of fuzzy graphs, Pattern Recognition Letters, 9, pp.159-162,1987.

[6] K.R. Bhuttani, A Rosenfield, Strong arcs in fuzzy graphs, Information Sciences, 152, pp.319-322,2003.

[7] Euler, Leonhard ,"Solutio problematis ad geometriam situs pertinentis", Comment. Acad. Sci. U. Petrop 8, pp.128–40,1736.

[8] K.P.Huber M.R.Berthold, Application of fuzzy graphs formet a modeling, Proceedings of the 2002 IEEE Conference,pp.640-644.

[9] A. Kiss, “An application of fuzzy graphs in database theory”, Pure Mathematics and Applications 1, pp.337-342, 1991.

[10] S. Mathew, M.S. Sunitha, “Types of arcs in a fuzzy graph”, Information Sciences, 179, pp.1760-1768,2009.

DOI: 10.5281/zenodo.3908341

[11] S. Mathew, M.S. Sunitha, "Node connectivity, arc connectivity of a fuzzy graph", Information Sciences, 180, pp.519-531,2010.

[12] J.N.mordeson,P.S.Nair, "Fuzzy Graphs and Fuzzy Hypergraphs",physica, verlag,Heidelberg,2000.

[13] J.N.Mordeson,C.S Peng,"Operations on fuzzy graphs",Information Sciences 79, pp.159-170, 1994.

[14] J.N.Moderson, "Fuzzy line graphs",Pattern Recognition Letters, 14, pp.381-384,1993.

[15] G. Pasi, R. Yager, K.T. Atanassov, "Intuitionistic fuzzy graph interpretations of multi-person multi-criteria decision making: generalized net approach", Intelligent Systems. Proceedings. 2004 2nd International IEEE Conference, pp.434-439, 2004.

[16] R. Parvathi, M.G. Karunambigai, "Intuitionistic fuzzy graphs", Journal of computational Intelligence: Theory and Applications, pp.139-150,2006.

[17] R. Parvathi, M.G. Karunambigai, K.T. Atanassov, "Operations on intuitionistic fuzzy graphs", Fuzzy Systems, 2009. FUZZ-IEEE 2009. IEEE International Conference, pp.1396-1401, 2009.

[18] A. Rosenfield, "fuzzy graphs",in: L.A.Zadeh,K.S.Fu,K.Tanaka,M.Shimura(Eds.), Fuzzy Sets and Their Applications to Cognitive and Decision Processes,Academic Press, New York, pp.77-95,1975.

[19] Said et al (2016). "Single valued neutrosophic graphs",Journal of New Theory,10, 86-101,2016.

[20] F.Smarandache, "Neutrosophic set - a generalization of the intuitionistic fuzzy set", Granular Computing, IEEE International Conference, 3842,2006, DOI: 10.1109/ GRC.2006.1635754.

[21] F. Smarandache,"A geometric interpretation of the neutrosophic set- A generalization of the intuitionistic fuzzy set", Granular computing (GRC),2011 IEEE International Conference, 602 606.DOI 10.1109/GRC.2011.6122665.

[22] F.Smarandache,"Symbolic Neutrosophic Theory", Europanovaasbl, Brussels, 195p, 2015.

[23] M.Mullai, "Domination in neutrosophic graphs", Neutrosophic Graph Theory and Algorithms, IGI Global Publications,pp.131-147, 2019.

DOI: 10.5281/zenodo.3908341

A short remark on Gödel incompleteness theorem and its self-referential paradox from Neutrosophic Logic perspective

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Abstract

It is known from history of mathematics, that Gödel submitted his two incompleteness theorems, which can be considered as one of hallmarks of modern mathematics in 20th century. Here we argue that Gödel incompleteness theorem and its self-referential paradox have not only put Hilbert's axiomatic program into question, but he also opened up the problem deep inside the then popular Aristotelian Logic. Although there were some attempts to go beyond Aristotelian binary logic, including by Lukasiewicz's *three-valued logic*, here we argue that the problem of self-referential paradox can be seen as reconcilable and solvable from *Neutrosophic Logic* perspective. **Motivation of this paper:** These authors are motivated to re-describe the self-referential paradox inherent in Godel incompleteness theorem. **Contribution:** This paper will show how Neutrosophic Logic offers a unique perspective and solution to Godel incompleteness theorem.

Keywords: Gödel incompleteness theorem, unprovability, undecidability, Neutrosophic Logic, Aristotelian Logic

1.Introduction

"*This statement is unprovable*." You can try to prove or disprove that particular statement, but indeed the statement is unprovable. That is how Gödel's incompleteness theorem began, see also [1], as in neutrosophic triplet: *proved, disproved, unprovable (indeterminate)*. Try also another statement: "*This statement is undecidable*." Sounds

DOI: 10.5281/zenodo.3908371

Received: January 17, 2019 Revised: April 04, 2019 Accepted: May 29, 2019

interesting? It is in the particular logic of our language, the problem of unprovability and undecidability belong to true problems of Hilbert's axiomatic program.

According to Padula, which can be rephrased as follows [6]:

> "Bertrand Russell and A. N. Whitehead's **Principia Mathematica** (1910–1913), in the future assigned as PM, contained a proof that the entire of arithmetic can be created based on set hypothesis. With it they wanted to demonstrate that all arithmetic is established on rationale. Kurt Gödel's confirmation (1931) of the '*inadequacy*' of formal frameworks, for example, PM is significant for some reasons. It is significant throughout the entire existence of arithmetic and for additional improvements in science, for example, the hypothesis of calculations and the hypothesis of formal frameworks which has prompted the advancement of PCs and scripting languages, and advances towards man-made consciousness; for the development of scientific evidence and confirmation hypothesis; and for the improvement of rationale as it is educated today. It is fascinating in light of the fact that to ace it a comprehension of language is as significant as information on science."

In literature, there are expository works on that theorem, which is dubbed as one of the hallmarks of 20^{th} century mathematics. Rebecca Goldstein [2], wrote which can be paraphrased as follows:

> The verification that was to turn into the "well known Incompleteness Proof" had clearly been cultivated the prior year, when Gödel was 23, and it was to be submitted in 1932 as his *Habilitationsschrift*, the last stage in the drawn out procedure of turning into an Austrian or German Dozent. It is one of the most surprising bits of numerical thinking at any point created, shocking both in the straightforwardness of its fundamental system and in the unpredictability of its subtleties, the meticulous making an interpretation of metamathematics into science by method of what has come to be called Gödel numbering. It is a completely requested mixing of a few layers of "voices," both scientific and metamathematical, contrast converging into symphonious harmonies at no other time heard. Music seems to give an especially adept similitude, which is the reason Ernest Nagel and James R. Newman in their great explicatory work, Gödel's Proof, portrayed the evidence as an "amazing intellectual symphony."

It is known, that the Neutrosophic Logic [8] is the only logic that can deal with the paradoxes, since a paradox P is a proposition that is true (its truth degree T = 1) and false (its false degree F = 1) in the same time, and as a consequence the paradox is also completely indeterminate (its indeterminate degree I = 1). Therefore, the Neutrosophic truth-values of the paradox is P(1, 1, 1), where $1+1+1 = 3 > 1$.

This paper will discuss, albeit shortly, on how Neutrosophic Logic can offer resolution to Gödel incompleteness theorem and its self-referential paradox.

2. Background: what is formal axiomatic program?

According to Steinmetz [3], which can be rephrased as follows:

DOI: 10.5281/zenodo.3908371

> "A formal system is, basically, a framework that has been expressly and totally characterized. At its most fundamental level a proper framework comprises of a plainly characterized language. The language is involved an assortment of images that speak to the most crude components of the language and are utilized to build the equations of the framework alongside a rundown of decides that characterize what comprises a grammatically all around shaped or semantically important recipe. In this way, the depiction of the conventional framework is distinctive relying upon whether the proper framework is built from a proof-hypothetical or a model-hypothetical point of view. …
>
> A proverbial framework is a framework that takes at least one recipes to be the maxims of the framework, which may possibly be a boundless number of equations if an adage diagram is utilized. The aphorisms of the framework are an assortment of recipes that are declared to be all around evident and from which the various genuine equations or hypotheses of the framework are gathered. In a proof-hypothetical framework the hypotheses of the framework are deductively demonstrated from the aphorisms of the framework or from recently demonstrated hypotheses. In a model-hypothetical framework the maxims of the framework characterize the substantial connections that exist between the articles that comprise the model of the framework and consequently the hypotheses of the framework are demonstrated dependent on what is valid for the items inside the model."

Into such a formal axiomatic program of Hilbert in early 1900, then came the young mathematician Gödel (see also [2][3][5]). What he did was to put the entire Hilbert's axiomatic program into question.

3. Discussion on *self-referential* paradox and a principle of *included middle*

Now, it is also possible to ask: how does Godel's incompleteness theorem give us a hint into what many physicists try to find: *The Ultimate Theory* or often dubbed as "TOE"? Ben-Yaacov wrote in his abstract, which can be rephrased as follows :

> "An extreme Universal hypothesis – a total hypothesis that accounts, by means of not many and basic first standards, for all the marvels previously watched and that will ever be watched – has been, and still is, the desire of most physicists and researchers. However, an essential rule that is encapsulated in the aftereffects of Gödel's deficiency hypotheses is that self-referencing prompts consistent conflict or disappointment, as in the liar oddity or Russell's conundrum. In physical speculations self-referencing essentially happens when it is understood that the eyewitness is likewise a member in the accomplished marvels – *we, people, are a piece of the Universe while watching it.* In this manner self-referencing, and thusly intelligent conflicts, are unavoidable, and any hypothesis claiming to be Universal will undoubtedly be inadequate."[4]

He also puts forth argument :

DOI: 10.5281/zenodo.3908371

> "Does Gödel's theorem apply to physics ? A common argument in favour of applying Gödel's theorem to physics, is, more or less, that "Gödel's theorem applies to arithmetics which is the basis of mathematics, physics uses mathematics, therefore Gödel's theorem applies to physics."

Although there are counterarguments of the above statement, many problems with the advanced theoretical physics in the last 30-40 years seem to suggest that such is true for Gödel's theorem. With overreliance on heavy abstraction and sophisticated higher-mathematics, it became so hard to get our feet back to *grounded (observed) realities* - as proponents of grounded approach would say [7].

Now, after admitting this problem, then what is the resolution?

At this point, the following section will cite on how Neutrosophic Logic provides solution to the excluded middle principle in Aristotelian logic. According to one of us (FS):

> "FS extended the *Law of Included Middle* [<A>, <nonA>, and a third value <T> which resolves their contradiction at another level of reality] to the *Law of Included Multiple-Middle* [<A>, <antiA>, and <neutA>, where <neutA> is split into a multitude of neutralities between <A> and <antiA>, such as <neut$_1$A>, <neut$_2$A>, etc.]. The <neutA> value (i.e. neutrality or indeterminacy related to <A>) actually comprises the included middle value. Also, he extended the *Principle of Dynamic Opposition* [opposition between <A> and <antiA>] to the *Principle of Dynamic Neutrosophic Opposition* [which means oppositions among <A>, <antiA>, and <neutA>].[1]

Therefore there are more possibilities, beyond just excluded middle principle.

To summarize this discussion:

Godel incompleteness theorem actually exposes the fundamental problem in Aristotelian logic.That is excluded middle principle. As we may know, there are certain cases where paradoxes and even self-referential paradoxes exist.

So, in NL theory, it is always possible to find intermediate or third way::

(Standpoint A) -- intermediate/paradox – (Standpoint B)

But in NL theory we see those paradoxes in a new way, without rejecting it outright.

For example:

"This statement is unprovable."

[1] See FS's bio: http://fs.unm.edu/FlorentinSmarandache.htm, also url: http://fs.unm.edu/LawIncludedMultiple-Middle.pdf

DOI: 10.5281/zenodo.3908371

Or

"*How do you decide between undecidability and unprovable?*"

These two statements make Aristotelian logic defunct, but not Neutrosophic logic.

The Neutrosophic Logic is the only logic that can deal with the paradoxes, since a paradox P is a proposition that is true (its truth degree T = 1) and false (its false degree F = 1) in the same time, and as a consequence the paradox is also completely indeterminate (its indeterminate degree I = 1). Therefore, the Neutrosophic truth-values of the paradox is P(1, 1, 1), where 1+1+1 = 3 > 1. No other logics allow the sum of its components to go over 1. *Self-Referential Paradoxes* have the same neutrosophic representation: T =1, F = 1, and I = 1.

4. Concluding remarks

This paper argues that Gödel incompleteness theorem and its self-referential paradox have not only put Hilbert's axiomatic program into question, but he also opened up the problem deep inside the then popular Aristotelian Logic. Although there were some attempts to go beyond Aristotelian binary logic, including Lukasiewicz's three-valued logic, here it is argued that the problem of self-referential paradox can be seen as reconcilable and solvable from Neutrosophic Logic perspective.

Summarizing, in Neutrosophic Logic, the Neutrosophic truth-values of the paradox is P(1, 1, 1), where 1+1+1 = 3 > 1. No other logics allow the sum of its components to go over 1. *Self-Referential Paradoxes* have the same neutrosophic representation: T =1, F = 1, and I = 1.

Hopefully this article will inspire further investigations.

References

[1] K. Gödel. UBER FORMAL UNENTSCHEIDBARE S¨ATZE DER "PRINCIPIA MATHEMATICA" UND VERWANDTER SYSTEME I, aus: "*Monatshefte fur Mathematik und Physik*" 38 (1931), 173-198.

[2] Rebecca Goldstein. *Incompleteness: The Proof and Paradox of Kurt Gödel*. New York: Atlas Books, 2005, pp. 156.

[3] Jason W. Steinmetz. *An Intuitively Complete Analysis of Gödel's Incompleteness*. arXiv: 1512.03667

[4] Uri Ben-Ya'acov. Gödel's incompleteness theorem and Universal physical theories. 2019 *J. Phys.: Conf. Ser.* 1391 012067. doi:10.1088/1742-6596/1391/1/012067

[5] Peter Smith. *An introduction to Gödel's theorems*. Second edition. Cambridge: Cambridge University Press, 2013.

[6] Janice Padula. The logical heart of a classic proof revisited: A guide to Gödel's 'incompleteness' theorems. *Australian Senior Mathematics Journal 25 (1) 2011.*

DOI: 10.5281/zenodo.3908371

[7] V. Christianto & F. Smarandache. Lost in Mathematics: The Perils of Post-Empirical Science & Their Resolution. In press

[8] F. Smarandache. A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability", American Research Press, Rehoboth, NM, 1999.

DOI: 10.5281/zenodo.3908371

n-Refined Neutrosophic Groups I

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Abstract

The aim of this paper is to define for the first time the concept of n-refined neutrosophic group. This work is devoted to study some elementary properties of n-refined neutrosophic groups and to establish the algebraic basis of this structure such as n-refined neutrosophic subgroups, n-refined neutrosophic homomorphisms, and n-refined neutrosophic isomorphisms.

Keywords: n-Refined neutrosophic group, n-refined neutrosophic subgroup, n-refined neutrosophic homomorphism.

1. Introduction

Smarandache [2013] extended the neutrosophic set to refined [n-valued] neutrosophic set, and to refined neutrosophic logic, and to refined neutrosophic probability, i.e. the truth value T is refined/split into types of sub-truths such as T1, T2, …, similarly indeterminacy I is refined/split into types of sub-indeterminacies I1, I2, …, and the falsehood F is refined/split into sub-falsehood F1, F2, … . [9]. Neutrosophy as a new trend in logical studies founded by F. Smarandache provided interesting methods to deal with classical algebraic structures. Many neutrosophical algebraic structures came to light such as neutrosophic groups, neutrosophic rings, and neutrosophic loops. See [1,2,4]. Many studies were applied using the idea of splitting (refining) the indeterminacy I into two components I_1, I_2 with a multiplication operation defined by the following: $I_1.I_1 = I_1, I_2.I_2 = I_2, I_1.I_2 = I_1$, like refined neutrosophic groups. See [3]. This idea was used in soft computing in [8].

In [5,6], Smarandache came with an interesting idea about splitting the indeterminacy I into n-components $I_1, I_2, \dots, I_n$, which refers to many different degrees of indeterminacy. In [7], Smarandache defined algebraic operations between refined neutrosophic numbers and refined neutrosophic sets. This refining idea will be very useful for us, since it helps with generalizing the concept of neutrosophic group.

In this work, we define the concept of n-refined neutrosophic group for the first time using formal multiplication between sub-indeterminacies. This multiplication is defined as follows:

$I_i.I_j = I_{\min(i,j)}$. For example $I_2.I_5 = I_2$.

Also, concepts such as n-refined neutrosophic subgroups, homomorphisms, and AH-subgroups will be introduced.

2. Preliminaries

In the following section, we recall some important and useful definitions about neutrosophic groups.

DOI: 10.5281/zenodo.3908405

Received: January 26, 2019 Revised: April 16, 2019 Accepted: May 13, 2019

Definition 2.1: [1,5]

Let (G,*) be a group . Then the neutrosophic group is generated by G and I under * denoted by

N(G)={$< G \cup I >, *$}.

I is called the indeterminate (neutrosophic element) with the property $I^2 = I$.

Definition 2.2: [1]

Let N(G) be a neutrosophic group and H be a neutrosophic subgroup, i.e (H contains a proper subgroup of G) of N(G). Then H is a neutrosophic normal subgroup of N(G) if $xH = Hx$ for all $x \in N(G)$.

Definition 2.3: [1]

Let N(G) be a neutrosophic group. Then the center of N(G) is denoted by C(N(G)), and defined

$C(N(G)) = \{x \in N(G); xy = yx \; \forall \; y \in N(G)\}$.

Definition 2.4: [1]

Let N(G) , N(H) be two neutrosophic groups, then $N(G) \times N(H) = \{(g,h); g \in N(G), h \in N(H)\}$.

Definition 2.5 :[5]

Let N(G) , N(H) be two neutrosophic groups and $\varphi: N(G) \to N(H)$ is called a neutrosophic homomorphism if it is a homomrphism between G , H and $\varphi(I) = I'$.

Where I' is the neutrosophic element of N(H).

If φ is a correspondence one-to-one it is called a neutrosophic isomorphism.

3. Main results

In this section, we dicsuss our concepts and construct some related examples to clarify the validity of them.

Definition 3.1:

Let $(G,*)$ be a group, we define the corresponding n-refined neutrosophic group $N_n(G)$ as follows:

$N_n(G) = (< G \cup \{I_1, \dots, I_n\} >, *) = \{(a_0, a_1 I_1, \dots, a_n I_n);\; a_i \in G\}$.

It is easy to see that $N_n(G)$ is closed under $*$, and it is a semi group but not a group since I_i has no inverse with respect to $*$ in general.

Remark 3.2:

If (G,+) is an additive abelian group, then addition on $N_n(G)$ can be described as follows:

Consider $x = (a_0, a_1 I_1, \dots, a_n I_n), y = (b_0, b_1 I_1, \dots, b_n I_n)$, we have

$x + y = (a_0 + b_0, [a_1 + b_1] I_1, \dots, [a_n + b_n] I_n)$. In this case $(N_n(G), +)$ is a classical abelian group.

DOI: 10.5281/zenodo.3908405

The identity element is (0,0,…,0).

It is easy to see that $N_n(G) \cong G \times G \times .. \times G \ (n+1 \ times)$ in the case of abelian additive group G.

Example 3.3:

Let $G = Z_2$ be the additive group of integers modulo 2, the corresponding 2-refined neutrosophic group is $N_2(G) = \{(0,0,0), (1,0,0), (0, I_1, 0), (0,0, I_2), (1, I_1, 0), (1,0, I_2), (0, I_1, I_2), (1, I_1, I_2)\}$.

We clarify addition as follows:

Remark 3.4:

If G is a multiplicative group, then group product on $N_n(G)$ can be described as follows:

Consider $x = (a_0, a_1 I_1, \dots, a_n I_n), y = (b_0, b_1 I_1, \dots, b_n I_n)$, we have

$xy = (t_0, t_1, \dots, t_n);\ t_s = \prod_{i,j=0}^{n} (a_i b_j) I_i I_j\ ;\ I_0 = e_G\ and\ I_i I_j = I_s$.

The identity element is $(e_G, e_G I_1, \dots, e_G I_n)$.

In this case $N_n(G)$ is not isomorphic to the direct product of n+1 copies of G, since it is not a classical group in this case.

Example 3.5:

Let $G = S_3 = \{f_0 = I, f_1, f_2, f_3, f_4, f_5\}$ be the non abelian symmetric group of order 6, the corresponding 2-refined neutrosophic group is $N_2(G) = \{(a, bI_1, cI_2);\ a, b, c \in G\}$.

We clarify the product on $N_2(G)$ as follows:

Consider $x = (f_1, f_2 I_1, f_0 I_2), y = (f_2, f_1 I_1, f_4 I_2)$, we have

$xy = (f_1 f_2, [(f_1 f_1)(f_2 f_2)(f_2 f_1)(f_2 f_4)(f_0 f_1)] I_1, [(f_1 f_4)(f_0 f_2)(f_0 f_4)] I_2)$.

The identity is $(f_0, f_0 I_1, f_0 I_2)$.

Definition 3.6:

(a) Let $N_n(G)$ be an n-refined neutrosophic group. It is called abelian if $x * y = y * x$ for all $x, y \in N_n(G)$.

(b) The subset $Z(N_n(G)) = \{y \in N_n(G); y * x = x * y\ for\ all\ x \in N_n(G)\}$ is called n-refined neutrosophic center.

Theorem 3.7:

Let $N_n(G)$ be an n-refined neutrosophic group. Then

(a) If G is abelian, $N_n(G)$ is abelian.

(b) $N_n(G)$ is abelian if and only if $N_n(G) = Z(N_n(G))$.

Proof:

The proof is similar to the classical case.

DOI: 10.5281/zenodo.3908405

Definition 3.8:

Let $N_n(G)$ be an n-refined neutrosophic group, H be a nonempty subset of $N_n(G)$, we say that H is an n-refined neutrosophic subgroup if H contains a subgroup of G.

Example 3.9:

Let $G = Z_2$ be the additive group of integers modulo 2, the corresponding 3-refined neutrosophic group is $N_3(G)$. The set $H = \{(0,0,0), (1,0,0), (1, I_1, 0), (1,0, I_2)\}$ is an n-refined neutrosophic subgroup of $N_3(G)$, since it contains $G = \{(0,0,0), (1,0,0)\}$ which is isomorphic to a subgroup of G. (We can consider it as a subgroup of G).

By previous example, we can see that Lagrange's theorem is not true in general in the case of finite n-refined neutrosophic group.

Definition 3.10:

Let $N_n(G)$ be an n-refined neutrosophic group, we denote to the number of elements in $N_n(G)$ by

O($N_n(G)$). If $N_n(G)$ is finite, then $O\big(N_n(G)\big) = m$, elsewhere $O\big(N_n(G)\big) = \infty$.

$O\big(N_n(G)\big)$ is called the order of n-refined neutrosophic group $N_n(G)$.

Theorem 3.11:

Let G be a finite group, $N_n(G)$ be its corresponding n-refined neutrosophic group. Then if $O(G) = m$, we have $O\big(N_n(G)\big) = m^{n+1}$.

Proof:

Since $N_n(G) = (< G \cup \{I_1, \dots, I_n\} >, *) = \{(a_0, a_1 I_1, \dots, a_n I_n);\ a_i \in G\}$, we find that

$O\big(N_n(G)\big) = O(G) \times O(G) \times \dots \times O(G)(n + 1\ times) = m^{n+1}$.

Definition 3.12:

Let $N_n(G), N_n(K)$ be two n-refined neutrosophic groups, $f: N_n(G) \to N_n(K)$ be a well defined map, we say that f is an n-refined neutrosophic homomorphism if:

(a) $f(xy) = f(x)f(y)$ for all $x, y \in N_n(G)$.

(b) $f(e_G, e_G, \dots, I_k, e_G, \dots, e_G) = (e_G, e_G, \dots, I_k, e_G, .., e_G)$.

Example 3.13:

Let $G = Z, K = Z_4, f: N_2(G) \to N_2(K); f(x, yI_1, zI_2) = ((x\ mod4), (y\ mod4)I_1, (z\ mod4)I_2)$

, where $x, y, z \in Z$.

Let $m = (x, yI_1, zI_2), n = (a, bI_1, cI_2)$ be two arbitrary elements in $N_2(G)$, it is clear that

$f(m + n) = f(m) + f(n)$.

DOI: 10.5281/zenodo.3908405

$f(I_1) = f(0,1.I_1, 0.I_2) = (0, I_1, 0), f(I_2) = f(0,0.I_1, 1.I_2) = (0,0, I_2)$. Thus f is an n-refined neutrosophic homomorphism.

Definition 3.14:

Let $N_n(G), N_n(K)$ be two n-refined neutrosophic groups, $f: N_n(G) \to N_n(K)$ be an n-refined neutrosophic homomorphism, we define:

(a) $Ker(f) = \{x \in N_n(G); f(x) = e_{N_n(K)}\}$.

(b) $Im(f) = \{y \in N_n(K);\ \exists\, x \in N_n(G): f(x) = y\}$.

Theorem 3.15:

Let $N_n(G), N_n(K)$ be two n-refined neutrosophic groups, $f: N_n(G) \to N_n(K)$ be an n-refined neutrosophic homomorphism, we have:

(a) $Ker(f)$ is a an n-refined neutrosophic subgroup of $N_n(G)$.

(b) $Im(f)$ is an n-refined neutrosophic subgroup of $N_n(K)$.

Proof:

(a) Since the restriction f_G of f is a homomorphism between

G and K, its kernel $Ker(f_G)$ will be a subset of $Ker(f)$, i.e $Ker(f)$ contains a subgroup of G, hence it is an n-refined neutrosophic subgroup according to Definition 3.8.

(b) The proof is similar to (a).

Example 3.16:

Let $G = Z, K = Z_4, f: N_2(G) \to N_2(K); f(x, yI_1, zI_2) = ((x\,mod4), (y\,mod4)I_1, (z\,mod4)I_2)$

, where $x, y, z \in Z$.

$Ker(f) = (4Z, 4ZI_1, 4ZI_2) = \{4x, 4yI_1, 4zI_2; x, y, z \in Z\}$, which is a 2-refined neutrosophic subgroup, since it contains $L = 4Z$.

$Im(f) = \{(a, bI_1, cI_2); a, b, c \in Z_4\} = N_2(K)$, which is a 2-refined subgroup, since it contains $S = Z_4$.

Definition 3.17:

Let $N_n(G), N_n(K)$ be two n-refined neutrosophic groups, $f: N_n(G) \to N_n(K)$ be an n-refined neutrosophic homomorphism, we call it an n-refined neutrosophic isomorphism if it is bijective.

Example 3.18:

Let $G = Z$ be the group of integers with normal addition, $N_3(G) = \{(a, bI_1, cI_2, dI_3); a, b, c, d \in Z\}$ be its corresponding 3-refined neutrosophic group. We define

$f: N_3(G) \to N_3(G); f(a, bI_1, cI_2, dI_3) = (-a, bI_1, cI_2, dI_3)$, it is clear that f is a bijective n-refined neutrosophic homomorphism, thus it is an n-refined neutrosophic isomorphism.

DOI: 10.5281/zenodo.3908405

Theorem 3.19:

Let $N_n(G)$ be an n-refined neutrosophic group. The set $GI_k = \{x * I_k; 1 \le k \le n\}$ has a group structure.

Proof:

We define the following operation on GI_k:

$(xI_k) \times (yI_k) = (x * y)I_k$. It is east to check that $\times$ is well defined, associative with I_k as an identity, and for each $xI_k \in GI_k$, there is an inverse $x^{-1}I_k$; x^{-1} is the inverse of x in G.

The previous group is called k-th pure neutrosophic component of $N_n(G)$.

Example 3.20:

Let $G = Z_3$ be the group of integers modulo 3 with respect to addition modulo 3,

$N_3(G) = \{(a, bI_1, cI_2, dI_3); a, b, c, d \in G\}$ be its corresponding 3-refined neutrosophic group.

The second pure neutrosophic subgroup of $N_3(G)$ is $G + I_2 = \{I_2, 1 + I_2, 2 + I_2\}$.

The group's binary operation on $G + I_2$ is defined as follows:

$(x + I_2) + (y + I_2) = (x + y) + I_2$. For all $x, y \in G$.

Theorem 3.21:

Let $N_n(G)$ be an n-refined neutrosophic group, GI_k be its k-th neutrosophic component. Then $G \cong GI_k$.

Proof:

Define $f: GI_k \to G; f(xI_k) = x$, f is a group isomorphism clearly, thus we get the proof.

Definition 3.22:

Let G, H be two groups, $G \times H$ be the corresponding direct product. We define the direct product of the related n-refined neutrosophic groups as follows:

$N_n(G) \times N_n(H) =< G \times H \cup \{I_1, .., I_n\} >$. We call it the n-refined neutrosophic direct product.

It is clear that $N_n(G) \times N_n(H)$ is an n-refined neutrosophic group, since $G \times H$ is a classical group.

Example 3.23:

Let $G = Z_2$ be the group of integers modulo 2 with respect to normal addition modulo 2, we construct the 2-refined neutrosophic direct product of G with itself.

$N_2(G) \times N_2(G) =< G \times G \cup \{I_1, I_2\} >= \{(a, bI_1, cI_2); a, b, c \in G \times G\}$.

We clarify addition on $N_2(G) \times N_2(G)$, consider $x = \big((1,0), (0,1)I_1, (1,1)I_2\big), y = ((0,0), (1,1)I_1, (0,1)I_2)$.

We have: $x + y = ((1,0), (1,0)I_1, (1,0)I_2)$.

DOI: 10.5281/zenodo.3908405

Definition 3.24:

Let $N_n(G) = \{(a_0, a_1I_1, \dots, a_nI_n);\ a_i \in G\}$ be an n-refined neutrosophic group,

$N_n(H) = \{(b_0, b_1I_1, \dots, b_nI_n);\ b_i \in H_i;\ H_i$ is a subgroup of $G\ for\ all\ i\}$ is called an AH-subgroup of $N_n(G)$.

If $H_i \cong H_j\ for\ all\ i \neq j$, then it is called an AHS-subgroup.

The AH-subgroup $N_n(H)$ is called AH-abelian if H_i is abelian for all i. Also, it is called AH-cyclic if H_i is cyclic for all i.

Example 3.25:

Let $G = S_3$ be the non abelian symmetric group of order 6, there are two non isomorphic subgroups of G,

$K \cong Z_2, S \cong Z_3$, consider the corresponding 3-refined neutrosophic group $N_3(G)$, we have:

$N_3(H) = (K, SI_1, KI_2, SI_3) = \{(a, bI_1, cI_2, dI_3); a, c \in K\ and\ b, d \in S\}$ is an AH-subgroup of $N_3(G)$.

$N_3(H)$ is an AH-cyclic, since K, S are cyclic.

5. Conclusion

In this article we have defined the concept of n-refined neutrosophic group for the first time. Also, we have introduced some corresponding notions such as n-refined neutrosophic subgroup, n-refined neutrosophic homomorphism, and n-refined neutrosophic isomorphism. Many examples were constructed to clarify these concepts.

Future researches

This work has established the theory of n-refined neutrosophic groups. In future works, we aim to define normality, quotients, and to study AH-substructures in n-refined neutrosophic groups.

Funding: "This research received no external funding"

Conflicts of Interest: "The authors declare no conflict of interest."

References

[1] Agboola, A.A.A,. Akwu, A.D,. and Oyebo, Y.T,. "Neutrosophic Groups and Subgroups", International .J .Math.Combin, Vol. 3, pp. 1-9, 2012.

[2] Agboola, A.A.A., Akinola, A.D., and Oyebola, O.Y.," NeutrosophicRings I ", International J.Mathcombin, Vol. 4, pp. 1-14, 2011.

[3] Agboola, A.A.A., "On Refined Neutrosophic Algebraic Structures", Neutrosophic Sets and Systems, Vol. 10, pp. 99-102, 2015.

[4] Kandasamy, V.W.B., and Smarandache, F., "Some Neutrosophic Algebraic Structures and Neutrosophic N-Algebraic Structures", Hexis, Phonex, Arizona 2006.

[5] Smarandache, F., "Symbolic Neutrosophic Theory", EuropaNova asbl, Bruxelles, 2015.

DOI: 10.5281/zenodo.3908405

[6] Smarandache, F., "*n-Valued Refined Neutrosophic Logic and Its Applications in Physics*", Progress in Physics, Vol. 4, pp.143-146, 2013.

[7] Smarandache, F., "Refined Literal Indeterminacy and the Multiplication Law of Sub-Indeterminacies", Neutrosophic Sets ans Systems, Vol. 9, 2015.

[8] Deli, I., " Refined Neutrosophic Sets and Refined Neutrosophic Soft Sets: Theory and Applications", Hand Book of Research on generalized and Hybrid Set Structures and Applications for Soft Computing, 2016.

[9] Smarandache, Florentin, n-Valued Refined Neutrosophic Logic and Its Applications in Physics, Progress in Physics, 143-146, Vol. 4, 2013; https://arxiv.org/ftp/arxiv/papers/1407/1407.1041.pdf
http://fs.unm.edu/n-ValuedNeutrosophicLogic-PiP.pdf

DOI: 10.5281/zenodo.3908405

On Neutrosophic Crisp Relations

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Abstract

The aim of this paper is to introduce a new types of neutrosophic crisp relations as a generalization to intuitionistic relations due to Indira et al.[9], and study some of its properties. Finally, the concepts of the star and retract neutrosophic relations are introduces and studied and some properties of these concepts will be investigated.

Keywords: Crisp sets relations; Neutrosophic crisp set; Star neutrosophic crisp set; Neutrosophic crisp relation; Star neutrosophic crisp relation.

1.Introduction

Established by Florentin Smarandache, neutrosophy [15] was presented as the study of origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. The main idea was to consider an entity,"A" in relation to its opposite"Non-A", and to that which is neither"A" nor "Non-A",denoted by "Neut-A". And from then on, neutrosophy became the basis of neutrosophic logic, neutrosophic probability, neutrosophic set, and neutrosophic statistics. According to this theory every idea "A" tends to be neutralized and balanced by "neutA" and "nonA" ideas - as a state of equilibrium. In a classical way"A","neutA", and"antiA" are disjoint two by two. Nevertheless, since in many cases the borders between notions are vague and imprecise, it is possible that "A", "neutA", and "antiA" have common parts two by two, or even all three of them as well. In [18, 19, 20], Smarandache introduced the fundamental concepts of neutrosophic set, that had led Salama et al. in [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17] to provide a mathematical treatment for the neutrosophic phenomena which already existed in our real world. Moreover, the work of Salama et al. formed a starting point to construct new branches of neutrosophic mathematics. Hence, neutrosophic set theory turned out to be a generalization of both the classical and fuzzy counterparts [2, 3, 6, 21]. This paper is devoted for introducing a new type of neutrosophic crisp relation called the retract neutrosophic crisp set, and studying some of its properties. On the application of neutrosophic theory, the readers can referes [22-24].

2. Preliminaries:

DOI: 10.5281/zenodo.3908413

Received: February 10, 2019 Revised: April 27, 2019 Accepted: June10, 2019

In this section, we recall some definitions for essential concepts of neutrosophic crisp relations on neutrosophic crisp sets and study their properties, which were introduced by Salama and Smarandache in [11,14].

2.1 Definition[11]

Consider any two neutrosophic crisp sets, A on X and B on Y; where $\mathrm{A} = \langle \mathrm{A}_1 , \mathrm{A}_2 , \mathrm{A}_3 \rangle$ and $\mathrm{B} = \langle \mathrm{B}_1 , \mathrm{B}_2 , \mathrm{B}_3 \rangle$

The Cartesian product of A and B is defined as the triple structure: $\mathrm{A} \times \mathrm{B} = \langle \mathrm{A}_1 \times \mathrm{B}_1 , \mathrm{A}_2 \times \mathrm{B}_2 , \mathrm{A}_3 \times \mathrm{B}_3 \rangle$.

where each component is a subset of the Cartesian product $\mathrm{X} \times \mathrm{Y}$; $\mathrm{A}_i \times \mathrm{B}_i = \{(a_i, b_i): a_i \in \mathrm{A}_i , b_i \in \mathrm{B}_i\}, \forall\, i = 1,2,3$

2.1 Corollary

In general if $\mathrm{A} \neq \mathrm{B}$, then $\mathrm{A} \times \mathrm{B} \neq \mathrm{B} \times \mathrm{A}$

2.2 Definition[11]

A neutrosophic crisp relation R from a neutrosophic crisp set A to B, namely $\mathrm{R} : \mathrm{A} \to \mathrm{B}$, is defined as a triple structure of the form $\mathrm{R} = \langle \mathrm{R}_1, \mathrm{R}_2, \mathrm{R}_3 \rangle$, where$\mathrm{R}_\mathrm{i} \subseteq \mathrm{A}_\mathrm{i} \times \mathrm{B}_\mathrm{i}, \forall\, \mathrm{i} = 1,2,3$ that is

2.3 Definition

A neutrosophic crisp inverse relation R^{-1} is a neutrosophic crisp relation from a neutrosophic crisp set B to A, $R^{-1}: B \to A$, and to be defined as a triple structure of the form: $R^{-1} = \langle R_1^{-1}, R_2^{-1}, R_3^{-1} \rangle$, where $R_i^{-1} \subseteq B_i \times A_i, \forall\, i = 1,2,3$ that is: $R_i^{-1} = \{(a_i, b_i) : (\mathrm{a}_i, \mathrm{b}_i) \in R_i\} R_i = \{(a_i, b_i): \mathrm{a_i} \in \mathrm{A_i} , \mathrm{b_i} \in \mathrm{B_i}\}$

2.1 Example

Let $\mathrm{X} = \{1, 2, 3,4 \}$, $\mathrm{A} = \langle \{1,2\}, \{3\}, \{4\} \rangle$ and $\mathrm{B} = \langle \{1\}, \{3\}, \{4,2\} \rangle$, if R is relation from A to B be defined as $\mathrm{R} = \{(a, b): a \in A, b \in B : a \geq b\}$, then the products of two neutrosophic crisp sets are given by:

$$A \times B = \langle \{(1,1), (2,1)\}, \{(3,3)\}, \{(4,4), (4,2)\} \rangle,$$
$$B \times A = \langle \{(1,1), (1,2)\}, \{(3,3)\}, \{(4,4), (2,4)\} \rangle,$$
$$R_1 = \langle \{(1,1)\}, \{(3,3)\}, \{(4,4)\} \rangle, R_1 \subseteq A \times B ,$$
$$R_2 = \langle \{(1,2)\}, \{(3,3)\}, \{(4,4), (2,4)\} \rangle, R_2 \subseteq B \times A ,$$
$$R_1^{-1} = \langle \{(1,1), (1,2)\}, \{(3,3)\}, \{(4,4), (2,4)\} \rangle,$$
$$R_2^{-1} = \langle \{(2,1)\}, \{(3,3)\}, \{(4,4), (4,2)\} \rangle.$$

3. Domain and Range of Neutrosophic Crisp Relations

For any neutrosophic crisp relation $\mathrm{R} : \mathrm{A} \to \mathrm{B}$, we define the following:

– The domain of R, is defined as: $Dom(R) = \langle dom(R_1), dom(R_2), dom(R_3) \rangle$

– The range of R, is defined as: $Rng(R) = \langle rng(R_1), rng(R_2), rng(R_3) \rangle$

– The Domain of R, is defined as: $Dom(R) = dom(R_1) \cup dom(R_2) \cup\, dom(R_3)$

DOI: 10.5281/zenodo.3908413

– The Range of R, is defined as: $Rng(R) = rng(R_1) \cup rng(R_2) \cup rng(R_3)$

3. 1 Corollary

From the definitions given in 2.5, one may notice that for any ultra neutrosophic crisp relation $\mathrm{R} : \mathrm{A} \to \mathrm{B}$, we have:

– The domain of R is a crisp subset of X, namely, $\mathrm{Dom(R)} \subseteq \mathrm{X}$.

–The range of R is a crisp subset of Y, namely, $Rng(R) \subseteq Y$.

3.2 Corollary

For any neutrosophic crisp relation $R : A \to B$, we have that:

$$Dom(R^{-1}) = Rng(R)$$

$$Rng(R^{-1}) = Dom(R)$$

4 Neutrosophic Crisp Relations' Operations

In the following definitions we consider R and S are two neutrosophic crisp relations between X and Y for every$(\mathrm{x}, \mathrm{y}) \in \mathrm{X} \times \mathrm{Y}$, neutrosophic crisp sets A and B in the form $\mathrm{A} = \langle \mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3 \rangle$ in X, $\mathrm{B} = \langle \mathrm{B}_1, \mathrm{B}_2, \mathrm{B}_3 \rangle$ on Y.

4.1 Definition [14]

The neutrosophic crisp relation R is a neutrosophic crisp subset of the neutrosophic crisp set S($\mathrm{R} \subseteq \mathrm{S}$), may be defined as one of the following two types:

Type 1: $\mathrm{R} \subseteq \mathrm{S} \iff A_{R1} \subseteq B_{S1}$, $A_{R2} \subseteq B_{S2}$ and$A_{R3} \supseteq B_{S3}$.

Type 2:$\mathrm{R} \subseteq \mathrm{S} \iff A_{R1} \subseteq B_{S1}$, $A_{R2} \supseteq B_{S2}$ and $A_{R3} \supseteq B_{S3}$.

4.2 Definition [14]

The neutrosophic intersection and neutrosophic union of any two neutrosophic sets A and B, may be defined as follows:

1. The neutrosophic intersection, $\mathrm{A} \cap \mathrm{B}$, may be defined as one of the following two types:

Type 1:$\mathrm{R} \cap \mathrm{S} \iff A_{R1} \cap B_{S1}$, $A_{R2} \cap B_{S2}$ and $A_{R3} \cup B_{S3}$.

Type 2: $\mathrm{R} \cap \mathrm{S} \iff A_{R1} \cap B_{S1}$, $A_{R2} \cup B_{S2}$ and $A_{R3} \cup B_{S3}$.

2. The neutrosophic intersection,$\mathrm{A} \cup \mathrm{B}$, may be defined as one of the following two types:

Type 1:$\mathrm{R} \cup \mathrm{S} \iff A_{R1} \cup B_{S1}$, $A_{R2} \cup B_{S2}$ and $A_{R3} \cap B_{S3}$.

Type 2:$\mathrm{R} \cup \mathrm{S} \iff A_{R1} \cup B_{S1}$, $A_{R2} \cap B_{S2}$ and $A_{R3} \cap B_{S3}$.

4.1 Theorem

Let R, S and Q be three neutrosophic crisp relations between X and Y for every$(\mathrm{x}, \mathrm{y}) \in \mathrm{X} \times \mathrm{Y}$, then:

a. $\mathrm{R} \subseteq \mathrm{S} \to \mathrm{R}^{-1} \subseteq \mathrm{S}^{-1}$.
b. $(\mathrm{R} \cup \mathrm{S})^{-1} \to \mathrm{R}^{-1} \cup \mathrm{S}^{-1}$, $(\mathrm{R} \cap \mathrm{S})^{-1} \to \mathrm{R}^{-1} \cap \mathrm{S}^{-1}$.
c. $(\mathrm{R}^{-1})^{-1} = \mathrm{R}$.
d. $\mathrm{R} \cap (\mathrm{S} \cup \mathrm{Q}) = (\mathrm{R} \cap \mathrm{S}) \cup (\mathrm{R} \cap \mathrm{Q}), \mathrm{R} \cup (\mathrm{S} \cup \mathrm{Q}) = (\mathrm{R} \cup \mathrm{S}) \cap (\mathrm{R} \cup \mathrm{Q})$.
e. If $\mathrm{S} \subseteq \mathrm{R}, \mathrm{Q} \subseteq \mathrm{R}$, then $\mathrm{S} \cup \mathrm{Q} \subseteq \mathrm{R}$.

4.3 Definition

DOI: 10.5281/zenodo.3908413

The neutrosophic crisp relationsR $\in \mathrm{NCR}(\mathrm{X}, \mathrm{X})$are called:

1) Neutrosophic Reflexive Relation, if for every $x \in X$, there is

$$(x, x) \in R_i \quad \forall\, i = 1,2,3$$

2) Neutrosophic Symmetric Relation, if $R = R^{-1}$, that is for every $(x, y) \in X \times Y$ such that

$$\forall\, (x, y) \in R_i \Rightarrow (y, x) \in R_i \quad \forall\, i = 1,2,3$$

3) Neutrosophic Transitive Relation, if there is$(x, y), (y, z) \in X \times Y$ such that

$$\forall\, (x, y), (y, z) \in R_i \Rightarrow (x, z) \in R_i \quad \forall\, i = 1,2,3$$

4) Neutrosophic Equivalence Relation, if R is reflexive, symmetric and transitive relations

5. Composition of Neutrosophic Crisp Relations

Consider the three neutrosophic crisp sets: A of X, B of Y and C of Z; and the two neutrosophic crisp relations:$R : A \to B$ and $S : B \to C$; where $R = \langle R_1, R_2, R_3 \rangle$, and $S = \langle S_1, S_2, S_3 \rangle$. The composition of R and S, is denoted and defined as:

$R \odot \mathrm{S} = \langle R_1 \circ S_1, R_2 \circ S_2, R_3 \circ S_3 \rangle$, such that:

$R_i \circ S_i : A_i \to C_i$, where $R_i \circ S_i = \{(a_i, b_i) : \exists \mathrm{b}_i \in B_i, (a_i, b_i) \in R_i \text{and} (b_i, c_i) \in S_i\}$.

5.1 Corollary

For any two neutrosophic crisp relations:$R : A \to B$ and $S : B \to C$;

$$Dom(R \odot \mathrm{S}) \subseteq Dom(R)$$

$$Rng(R \odot \mathrm{S}) \subseteq Rng(S)$$

5.2 Corollary

Consider the three neutrosophic crisp relations: $R : A \to B$ and $S : B \to C$, and

$$K : C \to D;$$

$$R \odot (S \odot \mathrm{K}) = (R \odot \mathrm{S}) \odot \mathrm{K}$$

6 Star Neutrosophic Crisp Relations

DOI: 10.5281/zenodo.3908413

In this section, we consider R^* and S^* are two star neutrosophic crisp relations between X and Y for every$(x, y) \in X \times Y$, star neutrosophic crisp sets A^* and B^*in the form $A^* = \langle A_1{}^*, A_2{}^*, A_3{}^* \rangle$ in X, $B^* = \langle B_1{}^*, B_2{}^*, B_3{}^* \rangle$ on Y.

6.1 Definition

Consider any two neutrosophic crisp sets, A on X and B on Y; where $A = \langle A_1, A_2, A_3 \rangle$ and $B = \langle B_1, B_2, B_3 \rangle$, two star neutrosophic crisp sets A^*, B^* is the structure $A^* = \langle A_1{}^*, A_2{}^*, A_3{}^* \rangle, B^* = \langle B_1{}^*, B_2{}^*, B_3{}^* \rangle$where $A_1{}^* = A_1 \cap co(A_2 \cup A_3)$, $A_2{}^* = A_2 \cap co(A_1 \cup A_3)$, $A_3{}^* = A_3 \cap co(A_1 \cup A_2), B_1{}^* = B_1 \cap co(B_2 \cup B_3)$, $B_2{}^* = B_2 \cap co(B_1 \cup B_3)$ and $B_3{}^* = B_3 \cap co(B_1 \cup B_2)$. Then:

The Cartesian product of A^* and B^* is defined as the triple structure:

$$A^* \times B^* = \langle A_1^* \times B_1^*, A_2^* \times B_2^*, A_3^* \times B_3^* \rangle.$$

where each component is a subset of the Cartesian product $X \times Y$;

$$A_i{}^* \times B_i{}^* = \{(a_i, b_i): a_i \in A_i{}^*, b_i \in B_i{}^*\}, \qquad \forall\, i = 1,2,3$$

6.2 Definition

A star neutrosophic crisp relation R^* from a star neutrosophic crisp set A^* to B^*, namely$R^* : A^* \to B^*$, is defined as a triple structure of the form $R^* = \langle R_1{}^*, R_2{}^*, R_3{}^* \rangle$, where $R_i^* \subseteq A_i^* \times B_i^*, \forall\, i = 1,2,3$, that is

$$R_i{}^* = \{(a_i, b_i): a_i \in A_i^*, b_i \in B_i^*\}$$

6.3 Definition

A star neutrosophic crisp inverse relation $R^{*-1}$is a star neutrosophic crisp relation from a neutrosophic crisp set B^* to A^*,$R^{*-1}: B^* \to A^*$, and to be defined as a triple structure of the form: $R^{*-1} = \langle R_1^{*-1}, R_2^{*-1}, R_3^{*-1} \rangle$, where $R_i^{*-1} \subseteq B_i^* \times A_i^*, \forall\, i = 1,2,3$ that is: $R_i^{*-1} = \{(a_i, b_i) : (a_i, b_i) \in R_i^{*-1}\}$

6.1 Example

Let$X = \{a, b, c, d, e, f\}$, $A = \langle \{a, b, c, d\}, \{e\}, \{f\} \rangle$ and $B = \langle \{a, b, c\}, \{d\}, \{e\} \rangle$, are neutrosophic crisp sets. Then $A^* = \langle \{a, b, c, d\}, \{e\}, \{f\} \rangle$, $B^* = \langle \{a, b, c\}, \{d\}, \{e\} \rangle$,then the products of two starneutrosophic crisp sets are given by:

$$A^* \times B^* = \langle \{(a,a),(a,b),(a,c)(b,a),(b,b),(b,c),(c,a),(c,b),(c,c),(d,a),(d,b),(d,c)\},\{(e,d)\},\{(f,e)\} \rangle,$$

$$B^* \times A^* = \langle \{(a,a),(a,b),(a,c),(a,d),(b,a),(b,b),(b,c),(b,d),(c,a),(c,b),(c,c),(c,d)\},\{(d,e)\},\{(e,f)\} \rangle,$$

$$R_1^* = \langle \{(a,a)\},\{(c,c)\},\{(d,d)\} \rangle, R_1 \subseteq A \times B,$$

$$R_2^* = \langle \{(a,b)\},\{(c,c)\},\{(d,d),(b,d)\} \rangle, R_2 \subseteq B \times A,$$

$$R_1^{*-1} = \langle \{(a,a),(a,b)\},\{(c,c)\},\{(d,d),(b,d)\} \rangle,$$

$$R_2^{*-1} = \langle \{(b,a)\},\{(c,c)\},\{(d,d),(d,b)\} \rangle.$$

7 .Domain and Range of Star Neutrosophic Crisp Relations

DOI: 10.5281/zenodo.3908413

For any neutrosophic crisp relation$R^* : A^* \to B^*$, we define the following:

– The domain of R^*, is defined as: $Dom(R^*) = \langle dom(R_1{}^*), dom(R_2{}^*), dom(R_3{}^*) \rangle$

– The range of R^*, is defined as: $Rng(R^*) = \langle Rng(R_1{}^{r}), Rng(R_2{}^{r}), Rng(R_3{}^{r}) \rangle$

– The Domain of R^*, is defined as: $Dom(R^*) = dom(R_1{}^*) \cup dom(R_2{}^*) \cup dom(R_3{}^*)$

– The Range of R^*, is defined as: $Rng(R^*) = rng(R_1{}^*) \cup rng(R_2{}^*) \cup rng(R_3{}^*)$

7.1 Corollary

From the definitions given in 3.5, one may notice that for any star neutrosophic crisp relation $R^* : A^* \to B^*$, we have:

– The domain of R^* is a crisp subset of X, namely, $\mathrm{Dom}(R^*) \subseteq X$.

– The range of R^* is a crisp subset of Y, namely, $Rng(R^*) \subseteq Y$.

7.2 Corollary

For any neutrosophic crisp relation $R^* : A^* \to B^*$, we have that: $Dom\left(R^{*-1}\right) = Rng(R^*),\ Rng\left(R^{*-1}\right) = Dom(R^*)$

8 Star Neutrosophic Crisp Relations' Operations

In this section, we call some definitions relations on star neutrosophic crisp sets and its properties.

8.1 Definition

The complement of a star neutrosophic crisp relation R^*(co R^*, for short) may be defined as:

$$co\ R^* = \langle R_3^*, R_2^*, R_1^* \rangle.$$

8.2 Definition

Consider R^* and S^* are two retract neutrosophic crisp relations between X and Y for every$(x,y) \in X \times Y$, retract neutrosophic crisp sets A^* and B^* in the form $A^* = \langle A_1{}^*, A_2{}^*, A_3{}^* \rangle$ in X, $B^* = \langle B_1{}^*, B_2{}^*, B_3{}^* \rangle$on Y.

1. The retract neutrosophic crisp relation R^{r} is a neutrosophic crisp subset of the retract neutrosophic crisp set S^* ($R^* \subseteq S^*$), may be defined as one of the following two types:

$$R^* \subseteq S^* \iff A^*_{R1} \subseteq B^*_{S1}, A^*_{R2} \subseteq B^*_{S2} \text{ and } A^*_{R3} \supseteq B^*_{S3}.$$

$$R^* \subseteq S^* \iff A^*_{R1} \subseteq B^*_{S1}, A^*_{R2} \supseteq B^*_{S2} \text{ and } A^*_{R3} \supseteq B^*_{S3}.$$

2. The neutrosophic intersection, $R^* \cap S^*$, of any two neutrosophic sets retract A^{r} andB^{r}, may be defined as follows:

$$R^* \cap S^* \iff A^*_{R1} \cap B^*_{S1}, A^*_{R2} \cap B^*_{S2} \text{ and } A^*_{R3} \cup B^*_{S3}.$$

$$R^* \cap S^* \iff A^*_{R1} \cap B^*_{S1}, A^*_{R2} \cup B^*_{S2} \text{ and } A^*_{R3} \cup B^*_{S3}.$$

3. The neutrosophic union, $R^* \cup S^*$, of any two neutrosophic sets retract A^{r} andB^{r}, may be defined as follows:

$$R^* \cup S^* \iff A^*_{R1} \cup B^*_{S1}, A^*_{R2} \cup B^*_{S2} \text{ and } A^*_{R3} \cap B^*_{S3}.$$

$$R^* \cup S^* \iff A^*_{R1} \cup B^*_{S1}, A^*_{R2} \cap B^*_{S2} \text{ and } A^*_{R3} \cap B^*_{S3}.$$

8.1 Theorem

Let $R^*, S^*$and Q^* be three retract neutrosophic crisp relations between X and Y for every$(x,y) \in X \times Y$, then:

a. $R^* \subseteq S^* \to R^{*-1} \subseteq S^{*-1}$.

b. $(R^* \cup S^*)^{-1} \to R^{*-1} \cup S^{*-1}$, $(R^* \cap S^*)^{-1} \to R^{*-1} \cap S^{*-1}$.

c. $(R^{*-1})^{-1} = R^*$.

d. $R^* \cap (S^* \cup Q^*) = (R^* \cap S^*) \cup (R^* \cap Q^*), R^* \cup (S^* \cup Q^*) = (R^* \cup S^*) \cap (R^* \cup Q^*)$.

DOI: 10.5281/zenodo.3908413

e. If $S^* \subseteq R^*$, $Q^* \subseteq R^*$, then $S^* \cup Q^* \subseteq R^*$.

9 Composition of Star Neutrosophic Crisp Relations

Consider the three neutrosophic crisp sets: A of X, B of Y and C of Z; and the two star neutrosophic crisp relations:$R^* : A^* \to B^*$ and $S^* : B^* \to C^*$; where

$R^* = \langle R_1{}^*, R_2{}^*, R_3{}^* \rangle$, and$S^* = \langle S_1{}^*, S_2{}^*, S_3{}^* \rangle$. The composition of R^* andS^*, is denoted and defined as: $R^* \odot S^* = \langle R_1{}^* \circ S_1{}^*, R_2{}^* \circ S_2{}^*, R_3{}^* \circ S_3{}^* \rangle$, such that: $R_i{}^* \circ S_i{}^* : A_i{}^* \to C_i{}^*$, where

$$R_i{}^* \circ S_i{}^* = \{(a_i, b_i) : \exists \mathrm{b}_i \in B_i{}^*, (a_i, b_i) \in R_i{}^* \text{and } (b_i, c_i) \in S_i{}^*\}.$$

9.1 Corollary

For any two star neutrosophic crisp relations: $R^* : A^* \to B^*$ and $S^* : B^* \to C^*$;

$$Dom(R^* \odot S^*) \subseteq Dom(R^*)$$

$$Rng(R^* \odot S^*) \subseteq Rng(S^*)$$

9.2 Corollary

Consider the three star neutrosophic crisp relations: $R^* : A^* \to B^*$ and$S^* : B^* \to C^*$, and $K^* : C^* \to D^*$;

$$R^* \odot (S^* \odot K^*) = (R^* \odot S^*) \odot K^*$$

9.1 Example

From the Example 6.1 is easy to get $R_i{}^* \circ S_i{}^*$, $Dom(R^* \odot S^*)$ and $Rng(R^* \odot S^*)$

10 Retract Neutrosophic Crisp Relations

In this section,we consider R^{r} and S^{r} are two retract neutrosophic crisp relations between X and Y for every$(\mathrm{x}, \mathrm{y}) \in \mathrm{X} \times \mathrm{Y}$, retract neutrosophic crisp sets A^{r} and B^{r}in the form $\mathrm{A}^{r} = \langle \mathrm{A}_1{}^{r}, \mathrm{A}_2{}^{r}, \mathrm{A}_3{}^{r} \rangle$ in X, $\mathrm{B}^{r} = \langle \mathrm{B}_1{}^{r}, \mathrm{B}_2{}^{r}, \mathrm{B}_3{}^{r} \rangle$ on Y.

10.1 Definition

Consider any two neutrosophic crisp sets, A on X and B on Y; where $\mathrm{A} = \langle \mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3 \rangle$ and $\mathrm{B} = \langle \mathrm{B}_1, \mathrm{B}_2, \mathrm{B}_3 \rangle$, two retract neutrosophic crisp sets $\mathrm{A}^{r}, \mathrm{B}^{r}$ is the structure $\mathrm{A}^{r} = \langle \mathrm{A}_1{}^{r}, \mathrm{A}_2{}^{r}, \mathrm{A}_3{}^{r} \rangle, \mathrm{B}^{r} = \langle \mathrm{B}_1{}^{r}, \mathrm{B}_2{}^{r}, \mathrm{B}_3{}^{r} \rangle$ where $\mathrm{A}_1{}^{r} = \mathrm{A}_1 \cap \mathrm{co}(\mathrm{A}_2 \cup \mathrm{A}_3)$, $\mathrm{A}_2{}^{r} = \mathrm{A}_2 \cap \mathrm{co}(\mathrm{A}_1 \cup \mathrm{A}_3)$, $\mathrm{A}_3{}^{r} = \mathrm{A}_3 \cap \mathrm{co}(\mathrm{A}_1 \cup \mathrm{A}_2)$, $\mathrm{B}_1{}^{r} = \mathrm{B}_1 \cap \mathrm{co}(\mathrm{B}_2 \cup \mathrm{B}_3)$, $\mathrm{B}_2{}^{r} = \mathrm{B}_2 \cap \mathrm{co}(\mathrm{B}_1 \cup \mathrm{B}_3)$ and $\mathrm{B}_3{}^{r} = \mathrm{B}_3 \cap \mathrm{co}(\mathrm{B}_1 \cup \mathrm{B}_2)$. Then:

The Cartesian product of A^{r} and B^{r} is defined as the triple structure:

$$\mathrm{A}^{r} \times \mathrm{B}^{r} = \langle \mathrm{A}_1^{r} \times \mathrm{B}_1^{r}, \mathrm{A}_2^{r} \times \mathrm{B}_2^{r}, \mathrm{A}_3^{r} \times \mathrm{B}_3^{r} \rangle.$$

DOI: 10.5281/zenodo.3908413

where each component is a subset of the Cartesian product X × Y;

$$A_i^{r} \times B_i^{r} = \{(a_i, b_i): a_i \in A_i^{r}, b_i \in B_i^{r}\}, \qquad \forall\, i = 1,2,3$$

10.2 Definition

A retract neutrosophic crisp relation R^{r} from a retract neutrosophic crisp set A^{r} to B^{r}, namely $R^{r} : A^{r} \to B^{r}$, is defined as a triple structure of the form $R^{r} = \langle R_1^{r}, R_2^{r}, R_3^{r}\rangle$, where $R_i^{r} \subseteq A_i^{r} \times B_i^{r}, \forall\, i = 1,2,3$, that is

$$R_i^{r} = \{(a_i, b_i): a_i \in A_i^{r}, b_i \in B_i^{r}\}$$

10.3 Definition

A retract neutrosophic crisp inverse relation $R^{r^{-1}}$ is a retract neutrosophic crisp relation from a neutrosophic crisp set B^{r} to A^{r}, $R^{-1}: B^{r} \to A^{r}$, and to be defined as a triple structure of the form: $R^{r^{-1}} = \langle R_1^{r^{-1}}, R_2^{r^{-1}}, R_3^{r^{-1}}\rangle$, where $R_i^{r^{-1}} \subseteq B_i^{r} \times A_i^{r}, \forall\, i = 1,2,3$ that is: $R_i^{r^{-1}} = \{(a_i, b_i) : (a_i, b_i) \in R_i^{r^{-1}}\}$

10.1 Example

Let $X = \{a, b, c, d\}$, $A = \langle \{a, b\}, \{c\}, \{d\} \rangle$ and $B = \langle \{a\}, \{c\}, \{b, d\} \rangle$, are neutrosophic crisp sets. Then $A^{r} = \langle\{a, b\}, \{c\}, \{d\}\rangle$, $B^{r} = \langle \{a\}, \{c\}, \{b, d\} \rangle$,then the products of two retractneutrosophic crisp sets are given by:

$$A^{r} \times B^{r} = \langle\{(a, a), (b, a)\}, \{(c, c)\}, \{(d, b), (d, d)\}\rangle,$$
$$B^{r} \times A^{r} = \langle\{(a, a), (a, b)\}, \{(c, c)\}, \{(b, d), (d, d)\}\rangle,$$
$$R_1^{r} = \langle\{(a, a)\}, \{(c, c)\}, \{(d, d)\}\rangle, R_1 \subseteq A \times B,$$
$$R_2^{r} = \langle\{(a, b)\}, \{(c, c)\}, \{(d, d), (b, d)\}\rangle, R_2 \subseteq B \times A,$$
$$R_1^{r-1} = \langle\{(a, a), (a, b)\}, \{(c, c)\}, \{(d, d), (b, d)\}\rangle,$$
$$R_2^{r-1} = \langle\{(b, a)\}, \{(c, c)\}, \{(d, d), (d, b)\}\rangle.$$

11 Domain and Range of Retract Neutrosophic Crisp Relations

For any neutrosophic crisp relation $R^{r} : A^{r} \to B^{r}$, we define the following:

– The domain of R^{r}, is defined as: $Dom(R^{r}) = \langle dom(R_1^{r}), dom(R_2^{r}), dom(R_3^{r})\rangle$

– The range of R^{r}, is defined as: $Rng(R^{r}) = \langle Rng(R_1^{r}), Rng(R_2^{r}), Rng(R_3^{r})\rangle$

– The Domain of R^{r}, is defined as: $Dom(R^{r}) = dom(R_1^{r}) \cup dom(R_2^{r}) \cup dom(R_3^{r})$

– The Range of R^{r}, is defined as: $Rng(R^{r}) = rng(R_1^{r}) \cup rng(R_2^{r}) \cup rng(R_3^{r})$

11.1 Corollary

DOI: 10.5281/zenodo.3908413

From the definitions given in 4.5, one may notice that for any retract neutrosophic crisp relation $R^{r} : A^{r} \rightarrow B^{r}$, we have:

– The domain of R^{r} is a crisp subset of X, namely, $\mathrm{Dom}(R^{r}) \subseteq X$.

– The range of R^{r} is a crisp subset of Y, namely, $Rng(R^{r}) \subseteq Y$.

11.2 Corollary

For any retract neutrosophic crisp relation $R^{r} : A^{r} \rightarrow B^{r}$, we have that: $Dom(R^{r^{-1}}) = Rng(R^{r})$,

$$Rng(R^{r^{-1}}) = Dom(R^{r})$$

11.1 Example

From the Example 10.1 is easy to get $\mathrm{Dom}(R^{r})$, $\mathrm{Rng}(R^{r})$,

12 Retract Neutrosophic Crisp Relations' Operations

In this section, we call some definitions relations on retract neutrosophic crisp sets and its properties.

12.1 Definition

The complement of a retract neutrosophic crisp relation R^{r} ($\mathrm{co}\ R^{r}$, for short) may be defined as:

$$r co\ R^{r} = \langle R_3^{r}, R_2^{r}, R_1^{r} \rangle.$$

12.2 Definition

Consider R^{r} and S^{r} are two retract neutrosophic crisp relations between X and Y for every$(x,y) \in X \times Y$, retract neutrosophic crisp sets A^{r} and B^{r} in the form $A^{r} = \langle A_1{}^{r}, A_2{}^{r}, A_3{}^{r} \rangle$ in X, $B^{r} = \langle B_1{}^{r}, B_2{}^{r}, B_3{}^{r} \rangle$ on Y.

4. The retract neutrosophic crisp relation R^{r} is a neutrosophic crisp subset of the retract neutrosophic crisp set S^{r} ($R^{r} \breve{\subseteq} S^{r}$), may be defined as one of the following two types:

$$R^{r} \breve{\subseteq} S^{r} \Leftrightarrow A^{r}{}_{R1} \subseteq B^{r}{}_{S1}, A^{r}{}_{R2} \subseteq B^{r}{}_{S2} \text{ and } A^{r}{}_{R3} \subseteq B^{r}{}_{S3}.$$

5. The neutrosophic intersection, $R^{r} \breve{\cap} S^{r}$, of any two neutrosophic sets retract A^{r} andB^{r}, may be defined as follows:

$$R^{r} \breve{\cap} S^{r} \Leftrightarrow A^{r}{}_{R1} \cap B^{r}{}_{S1}, A^{r}{}_{R2} \cap B^{r}{}_{S2} \text{ and } A^{r}{}_{R3} \cap B^{r}{}_{S3}.$$

6. The neutrosophic union, $R^{r} \breve{\cup} S^{r}$, of any two neutrosophic sets retract A^{r} andB^{r}, may be defined as follows:

$$R^{r} \breve{\cup} S^{r} \Leftrightarrow A^{r}{}_{R1} \cup B^{r}{}_{S1}, A^{r}{}_{R2} \cup B^{r}{}_{S2} \text{ and } A^{r}{}_{R3} \cup B^{r}{}_{S3}.$$

12. 1 Theorem

Let R^{r}, S^{r} and Q^{r} be three retract neutrosophic crisp relations between X and Y for every$(x,y) \in X \times Y$, then:

a) $R^{r} \breve{\subseteq} S^{r} \rightarrow R^{r^{-1}} \breve{\subseteq} S^{r^{-1}}$.

b) $(R^{r} \breve{\cup} S^{r})^{-1} \rightarrow R^{r^{-1}} \breve{\cup} S^{r^{-1}}$, $(R^{r} \breve{\cap} S^{r})^{-1} \rightarrow R^{r^{-1}} \breve{\cap} S^{r^{-1}}$.

DOI: 10.5281/zenodo.3908413

c) $(R^{r^{-1}})^{-1} = R^{r}$.

d) $R^{r} \cap (S^{r} \breve{\cup} Q^{r}) = (R^{r} \breve{\cap} S^{r}) \breve{\cup} (R^{r} \breve{\cap} Q^{r}), R \breve{\cup} (S^{r} \breve{\cup} Q^{r}) = (R^{r} \breve{\cup} S^{r}) \cap (R^{r} \breve{\cup} Q^{r})$.

e) If $S^{r} \breve{\subseteq} R^{r}$, $Q^{r} \breve{\subseteq} R^{r}$, then $S^{r} \breve{\cup} Q^{r} \subseteq R^{r}$.

13 Composition of Neutrosophic Crisp Relations

Consider the three neutrosophic crisp sets: A of X, B of Y and C of Z; and the two retract neutrosophic crisp relations: $R^{r} : A^{r} \rightarrow B^{r}$ and $S^{r} : B^{r} \rightarrow C^{r}$; where $R^{r} = \langle R_1{}^{r}, R_2{}^{r}, R_3{}^{r} \rangle$, and $S^{r} = \langle S_1{}^{r}, S_2{}^{r}, S_3{}^{r} \rangle$. The composition of R^{r} andS^{r}, is denoted and defined as: $R^{r} \odot S^{r} = \langle R_1{}^{r} \circ S_1{}^{r}, R_2{}^{r} \circ S_2{}^{r}, R_3{}^{r} \circ S_3{}^{r} \rangle$, such that: $R_i{}^{r} \circ S_i{}^{r} : A_i{}^{r} \rightarrow C_i{}^{r}$, Where $R_i{}^{r} \circ S_i{}^{r} = \{(a_i, b_i) : \exists b_i \in B_i{}^{r}, (a_i, b_i) \in R_i{}^{r} \text{and} (b_i, c_i) \in S_i{}^{r}\}$.

13.1 Corollary

For any two retract neutrosophic crisp relations: $R^{r} : A^{r} \rightarrow B^{r}$ and$S^{r} : B^{r} \rightarrow C^{r}$;

$$Dom(R^{r} \odot S^{r}) \subseteq Dom(R^{r})$$

$$Rng(R^{r} \odot S^{r}) \subseteq Rng(S^{r})$$

13.2 Corollary

Consider the three retract neutrosophic crisp relations: $R^{r} : A^{r} \rightarrow B^{r}$ and$S^{r} : B^{r} \rightarrow C^{r}$, and $K^{r} : C^{r} \rightarrow D^{r}$;

$$R^{r} \odot (S^{r} \odot K^{r}) = (R^{r} \odot S^{r}) \odot K^{r}$$

13.1 Example

From the Example 10.1 is easy to get Dom(R^{r}), Rng(R^{r}), and $Dom(R^{r} \odot S^{r})$

Conclusion

In this work, the concepts of star neutrosophic crisp relations and retract neutrosophic crisp relations were introduced. Added to, we have generalized the notion of crisp relation. Also, the main properties related to the neutrosophic crisp relations have been studied. Future work will be directed to study the notion of the neutrosophic crisp mapping for other types of relations based on neutrosophic crisp sets.

References

[1] Alblowi, S., Salama, A. A., and Eisa, M. New concepts of neutrosophic sets. International Journal of Mathematicsand Computer Applications Research (IJMCAR), vol. 4, no. 1, pp.59 – 66, 2014.

[2] Atanassov,K. intuitionistic fuzzy sets, in V.Sgurev,ed., Vii ITKRS Session, Sofia (June 1983 central Sci.andTechn. Library, Bulg. Academy of Sciences, 1984.

[3] Atanassov,K. intuitionistic fuzzy sets, Fuzzy Sets and Systems 20, pp.87-96, 1986.

[4] Hanafy, I., Salama, A. A., Mahfouz, K. "Neutrosophic classical events and its probability", International Journal of Mathematics and Computer Applications Research(IJMCAR) vol. 3, no. 3, pp.171- 178, 2013.

DOI: 10.5281/zenodo.3908413

[5] Salama, A. A., Alblowi, S. "Neutrosophic set and neutrosophic topological spaces", ISOR J. Mathematics vol. 3, no. 3, pp.31 – 35, 2012.

[6] Salama, A. A., and Alblowi, S. "Intuitionistic Fuzzy Ideals Topological Spaces", Advances in Fuzzy Mathematics , Vol.7(1), pp 51- 60, 2012.

[7] Salama, A. A., Khaled, O. M., and Mahfouz, K. "Neutrosophic correlation and simple linear regression", NeutrosophicSets and Systems vol. 5, pp.3 – 8, 2014.

[8] Salama, A. A. "Neutrosophic crisp point & neutrosophic crisp ideals", Neutrosophic Sets and Systems vol. 1, no.1, pp.50 – 54, 2013.

[9] Indira, S., Raja, Rajeswari, R., "A Study on Star Intuitionistic Sets", International Journal of Mathematics and Statistics Invention, (2)4, pp51-63, 2014.

[10] Salama, A. A., and Alblowi, S. A."Generalized Neutrosophic Set and Generalized Neutrosophic Spaces", Journal Computer Sci. Engineering, Vol. (2) No. (7), pp. 129-132, 2012

[11] Salama, A. A., Alblowi, S. A., Smarandache, F."Neutrosophic crisp open set and neutrosophic crisp continuity via neutrosophic crisp ideals". I.J. Information Engineering and Electronic Business vol. 3, pp.1 – 8, 2014.

[12] Salama, A. A., Elghawalby, H. "*- Neutrosophic Crisp Set & Relations", Neutrosophic Sets and Systems, Vol. 6, pp. 12-17, 2014.

[13] Salama, A. A., Alblowi, S. A., and Smarandache, F. "The characteristic function of a neutrosophic set", Neutrosophic Sets and Systems vol. 3, pp.14 – 18, 2014.

[14] Salama, A. A., Smarandache, F. "Neutrosophic Crisp Set Theory", Educational Publisher, Columbus, Ohio,USA., 2015.

[15] Salama, A. A., Smarandache, F, "Filters via Neutrosophic Crisp Sets", Neutrosophic Sets and Systems, Vol. 1, No. 1, pp. 34-38, 2013,

[16] Salama, A. A., Smarandache, F., Valeri Kroumov "Neutrosophic crisp Sets & Neutrosophic crisp Topological Spaces", Bulletin of the Research Institute of Technology, Okayama University of Science, Japan, in January-February 2014.

[17] Salama, A. A., Mohamed Eisa , M. M. Abdelmoghny. "Neutrosophic Relations Database", International Journal of Information Science and Intelligent System, 3(2), pp.33-46, 2014.

[18] Smarandache, F. "A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability", American Research Press, Rehoboth, NM, 1999.

[19] Smarandache, F. "Neutrosophy and neutrosophic logic", In First International Conference on Neutrosophy , Neutrosophic Logic, Set, Probability, and Statistics University of New Mexico, Gallup, NM 87301, USA 2002.

[20] Smarandache, F. "Neutrosophic set, a generialization of the intuitionistic fuzzy sets", Inter. J. Pure Appl.Math. vol. 24 ,pp.287-297, 2005.

[21] Zadeh, L.A. "Fuzzy Sets", Inform and Control 8, pp.338-353, 1965.

DOI: 10.5281/zenodo.3908413

[22] Broumi S., Bakali A., Talea M,, and Smarandache F,"Isolated Single Valued Neutrosophic Graphs", Neutrosophic Sets and Systems, Vol. 11, pp.74-78, 2016.

[23] Broumi S., Dey A., Bakali A., Talea M., Smarandache F., Son L. H., Koley D., "Uniform Single Valued Neutrosophic Graphs", Neutrosophic Sets and Systems, Vol. 17, pp.42-49, 2017.

[24] Broumi S., Son L.H., Bakali A., Talea M., Smarandache F., Selvachandran G., "Computing Operational Matrices in Neutrosophic Environments: A Matlab Toolbox", Neutrosophic Sets and Systems, Vol. 18, pp.58-66, 2017.

DOI: 10.5281/zenodo.3908413

n-Refined Neutrosophic Groups II

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Abstract

The objective of this paper is to study some of AH-substructures in n-refined neutrosophic group. Also, it deals with some elementary properties of AH-subgroups, AH-normality, AH-homomorphisms, and endomorphisms especially in a non abelian n-refined neutrosophic group.

Keywords: n-Refined neutrosophic group, AH- subgroup, AH- homomorphism, AH-endomorphism.

1. Introduction

Neutrosophy as a new kind of logic founded by F. Smarandache deals with indeterminacy in nature and reality. In particular, it provided a strong tool to study some algebraic structures. Many neutrosophical algebraic structures came to light such as neutrosophic groups, neutrosophic rings, and neutrosophic semi groups. See [2,3,6]. Many studies were carried out using the idea of splitting the indeterminacy I into two components I_1, I_2, like refined neutrosophic groups. See [3]. It has some interesting applications in soft computing [10].

In [7,8], F. Smarandache came with a new idea which suggests the splitting of indeterminacy I into n sub-indeterminacies $I_1, I_2, \dots, I_n$, which refers to many different degrees of indeterminacy. In [9], Smarandache defined algebraic operations between refined neutrosophic numbers and refined neutrosophic sets.

In [1], Abobala has defined the concept of n-refined neutrosophic groups to generalize the classical concept of neutrosophic group. Some interesting notions were presented such as n-refined neutrosophic subgroup, n-refined neutrosophic homomorphism, and AH-subgroup of an n-refined neutrosophic group.

In this work, we continue the work began in [1]. We establish the concept of AH-homomorphism to study AH-subgroups. Also, we handle some related concepts such as AH-nilpotency, AH-solvability, and AH-homomorphisms.

Motivation

This article is a continuation work of the work began in [1]. It establishes the algebraic theory of some AH-substructures in an n-refined neutrosophic ring.

2. Preliminaries

In the following section, we recall some important and useful definitions about neutrosophic groups and n-refined neutrosophic groups.

Definition 2.1: [2]

DOI: 10.5281/zenodo.3929792

Received: Feburary 18, 2019 Revised: May 18, 2019 Accepted: June 11, 2019

Let (G,*) be a group . Then the neutrosophic group is generated by G and I under * denoted by

N(G)={$< G \cup I >, *$}.

I is called the indeterminate (neutrosophic element) with the property $I^2 = I$.

Definition 2.2: [1]

Let $(G,*)$ be a group, we define the corresponding n-refined neutrosophic group $N_n(G)$ as follows:

$N_n(G) = (< G \cup \{I_1, \dots, I_n\} >, *) = \{(a_0, a_1 I_1, \dots, a_n I_n);\ a_i \in G\}$.

It is easy to see that $N_n(G)$ is closed under $*$, and it is a semi group but not a group since I_i has no inverse with respect to $*$ in general.

Remark 2.3: [1]

If (G,+) is an additive abelian group, then addition on $N_n(G)$ can be described as follows:

Consider $x = (a_0, a_1 I_1, \dots, a_n I_n), y = (b_0, b_1 I_1, \dots, b_n I_n)$, we have

$x + y = (a_0 + b_0, [a_1 + b_1]I_1, \dots, [a_n + b_n]I_n)$. In this case $(N_n(G), +)$ is a classical abelian group.

The identity element is (0,0,…,0).

It is easy to see that $N_n(G) \cong G \times G \times .. \times G\ (n + 1\ times)$ in the case of abelian additive group G.

Remark 2.4: [1]

If G is a multiplicative group, then group product on $N_n(G)$ can be described as follows:

Consider $x = (a_0, a_1 I_1, \dots, a_n I_n), y = (b_0, b_1 I_1, \dots, b_n I_n)$, we have

$xy = (t_0, t_1, \dots, t_n);\ t_s = \prod_{i,j=0}^{n} (a_i b_j) I_i I_j\ ;\ I_0 = e_G\ and\ I_i I_j = I_s$.

The identity element is $(e_G, e_G I_1, \dots, e_G I_n)$.

In this case $N_n(G)$ is not isomorphic to the direct product of n+1 copies of G, since it is not a classical group in this case.

The binary operation between the sub-indeterminacies is $I_i . I_j = I_{\min(i,j)}$.

Definition 2.5: [1]

Let $N_n(G) = \{(a_0, a_1 I_1, \dots, a_n I_n);\ a_i \in G\}$ be an n-refined neutrosophic group,

$N_n(H) = \{(b_0, b_1 I_1, \dots, b_n I_n);\ b_i \in H_i;\ H_i$ is a subgroup of $G\ for\ all\ i\}$ is called an AH-subgroup of $N_n(G)$.

If $H_i \cong H_j\ for\ all\ i \neq j$, then it is called an AHS-subgroup.

The AH-subgroup $N_n(H)$ is called AH-abelian if H_i is abelian for all i. Also, it is called AH-cyclic if H_i is cyclic for all i.

Example 2.6: [1]

Let $G = S_3$ be the non abelian symmetric group of order 6, there are two non isomorphic subgroups of G, $K \cong Z_2, S \cong Z_3$, consider the corresponding 3-refined neutrosophic group $N_3(G)$, we have:

$N_3(H) = (K, SI_1, KI_2, SI_3) = \{(a, bI_1, cI_2, dI_3); a, c \in K \ and \ b, d \in S\}$ is an AH-subgroup of $N_3(G)$.

$N_3(H)$ is an AH-cyclic, since K, S are cyclic.

Definition 2.7: [5]

Let G be a group with the following normal series $\{e\} = H_0 \leq H_1 \leq \cdots \leq H_n = G$. It is called solvable if H_i/H_{i-1} is abelian.

The intersection, direct product, and product of two solvable groups is solvable.

Definition 2.8: [11]

(a) Let G be any group. It is called meta abelian if it has a abelian derivative subgroup G'.

(b) Let G be a group. It is called nilpotent if it has a central series.

For the concept of central series, see [11].

S_3 is a solvable group, but it is not nilpotent.

D_4 is a meta abelian and nilpotent group.

The intersection, and the direct product of two meta abelian groups is meta abelian.

The intersection, and the direct product of two nilpotent groups is nilpotent.

Definition 2.9: [2]

Let N(G) be a neutrosophic group and H be a neutrosophic subgroup, i.e (H contains a proper subgroup of G) of N(G). Then H is a neutrosophic normal subgroup of N(G) if $xH = Hx$ for all $x \in N(G)$.

Definition 2.10: [2]

Let N(G) be a neutrosophic group. Then the center of N(G) is denoted by C(N(G)), and defined

$C(N(G)) = \{x \in N(G); xy = yx \ \forall \ y \in N(G)\}$.

Definition 2.11: [2]

Let N(G) , N(H) be two neutrosophic groups, then $N(G) \times N(H) = \{(g, h); g \in N(G), h \in N(H)\}$.

Definition 2.12: [6]

Let N(G) , N(H) be two neutrosophic groups and $\varphi: N(G) \to N(H)$ is called a neutrosophic homomorphism if it is a homomrphism between G , H and $\varphi(I) = I'$.

Where I' is the neutrosophic element of N(H).

DOI: 10.5281/zenodo.3929792

If φ is a correspondence one-to-one it is called a neutrosophic isomorphism.

3. Main discussion

Definition 3.1:

Let G be any group, $N_n(G)$ be its corresponding n-refined neutrosophic group,

$N_n(H) = (H_0, H_1I_1, \dots, H_nI_n);\ H_i \leq G$ be an AH-subgroup of $N_n(G)$. We say

(a) $N_n(H)$ is AH-normal subgroup if H_i is normal for all i.

(b) $N_n(H)$ is AH-nilpotent subgroup if H_i is nilpotent for all i.

(c) $N_n(H)$ is AH-solvable subgroup if H_i is solvable for all i.

(d) $N_n(H)$ is AH-meta abelian subgroup if H_i is meta abelian for all i.

(e) $N_n(H)$ is AH-simple subgroup if H_i is simplefor all i.

Example 3.2:

Consider the symmetric group of order 6 ($G = S_3$), it has one normal subgroup $H \cong Z_3$, and three 2-Sylow subgroups $K \cong S \cong L \cong Z_2$.

Let $N_2(G)$ be the corresponding 2-refined neutrosophic group, we have

(a) $M_1 = (K, KI_1, HI_2)$ is an AH-subgroup of $N_2(G)$.

(b) M_1 is an AH-nilpotent/AH-solvable, since K,H are nilpotent, and solvable subgroups of G.

(c) M_1 is an AH-simple, since K,H are simple.

(d) M_1 is not an AH-normal, since K is not normal.

(e) M_1 is an AH-meta abelian, since H,K are meta abelian subgroups.

Remark 3.3:

If G is an additive abelian group, then any AH-subgroup is a classical subgroup, since $N_n(G)$ is a classical group and isomorphic to the direct product of G with itself (n+1 times).

Definition 3.4:

Let G,K be any two groups, $N_n(G), N_n(K)$ be their corresponding n-refined neutrosophic groups,

$f_i: G \to K; 0 \leq i \leq n$ be a classical homomorphism for all i. We say

(a) $f: N_n(G) \to N_n(K); f(a_0, a_1I_1, .., a_nI_n) = (f_0(a_0), f_1(a_1)I_1, \dots, f_n(a_n)I_n)$ is an AH-homomorphism.

(b) If $f_i: G \to K$ is an isomorphism for all i, then $f: N_n(G) \to N_n(K)$ is called an AH-isomorphism.

(c) If $f_i = f_j; i \neq j$, then $f: N_n(G) \to N_n(K)$ is called an AHS-homomorphism.

DOI: 10.5281/zenodo.3929792

(d) The AH-kernel is defined as follows: $AH - Ker(f) = (Ker(f_0), Ker(f_1)I_1, .., Ker(f_n)I_n)$.

(e) The AH-image is defined as: $AH - Im(f) = (f_0(G), f_1(G)I_1, \dots, f_n(G)I_n)$.

We denote to the AH-homomorphism : $N_n(G) \to N_n(K); f(a_0, a_1I_1, .., a_nI_n) = (f_0(a_0), f_1(a_1)I_1, \dots, f_n(a_n)I_n)$ by $f = (f_0, f_1I_1, .., f_nI_n)$.

Definition 3.5:

Let G be any group, $N_n(G)$ be its corresponding n-refined neutrosophic group,

$N_n(H) = (H_0, H_1I_1, \dots, H_nI_n), N(K) = (K_0, K_1I_1, \dots, K_nI_n);\ H_i, K_i \leq G$ be any two AH-subgroups of $N_n(G)$.

(a) We define the intersection as follows: $N_n(H) \cap N_n(K) = (H_0 \cap K_0, (H_1 \cap K_1)I_1, .., (H_n \cap K_n)I_n)$.

(b) We define the product as follows: $N_n(H). N_n(K) = (H_0. K_0, (H_1. K_1)I_1, .., (H_n. K_n)I_n)$.

(c) We define the direct product as follows: $N_n(H) \times N_n(K) = (H_0 \times K_0, (H_1 \times K_1)I_1, .., (H_n \times K_n)I_n)$.

Example 3.6:

Consider the symmetric group of order 6 ($G = S_3$), it has one normal subgroup $H \cong Z_3$, and three 2-Sylow subgroups $K \cong S \cong L \cong Z_2$.

Let $N_2(G)$ be the corresponding 2-refined neutrosophic group, $M_1 = (K, KI_1, HI_2), M_2 = (H, KI_1, KI_2)$ be two AH-subgroups of $N_2(G)$, we have

(a) $M_1 \cap M_2 = (\{e\}, KI_1, \{e\}I_2)$, which is an AH-subgroup of $N_2(G)$.

(b) $M_1. M_2 = (KH, KKI_1, HKI_2) = (G, KI_1, GI_2)$.

Example 3.7:

Let $G = (Z, +), K = (Z_6, +)$ be two groups, $N_2(G), N_2(K)$ be the corresponding 2-refined neutrosophic groups, we have

(a) $f_0: G \to K;\ f_0(a) = a\ mod6, f_1: G \to K;\ f_1(a) = 2a\ mod6$ are two classical homomorphisms.

(b) $f: N_2(G) \to N_2(K); f(a_0, a_1I_1, a_2I_2) = (f_0(a_0), f_0(a_1)I_1, f_1(a_2)I_2) = (a_0\ mod6, a_1\ mod6\ I_1, 2a_2\ mod6\ I_2)$ is an AH-homomorphism.

(c) $AH - Ker(f) = (Ker(f_0), Ker(f_0)I_1, Ker(f_1)I_2) = (6Z, 6ZI_1, 3ZI_2)$.

(d) $AH - Im(f) = (Im(f_0), Im(f_0)I_1, Im(f_1)I_2) = (Z_6, Z_6I_1, \{0,2,4\}I_2)$.

Theorem 3.8:

Let G be any group, $N_n(G)$ be its corresponding n-refined neutrosophic group,

$N_n(H) = (H_0, H_1I_1, \dots, H_nI_n), N(K) = (K_0, K_1I_1, \dots, K_nI_n);\ H_i, K_i \leq G$ be any two AH-subgroups of $N_n(G)$. We have:

(a) $N_n(H) \cap N_n(K)$ is an AH-subgroup of $N_n(G)$.

DOI: 10.5281/zenodo.3929792

(b) If $N_n(H), N_n(K)$ are AH-normal, then $N_n(H).N_n(K)$ is AH-normal.

(c) $N_n(H) \times N_n(K)$ is an AH-subgroup of $N_n(G) \times N_n(G)$.

(d) If $N_n(H), N_n(K)$ are AH-abelian, then $N_n(H) \cap N_n(K), N_n(G) \times N_n(G)$ are AH-abelian.

(e) If $N_n(H), N_n(K)$ are AH-cyclic, then $N_n(H) \cap N_n(K)$ is AH-cycilc.

(f) If $N_n(H), N_n(K)$ are AH-nilpotent, then $N_n(H) \cap N_n(K), N_n(G) \times N_n(G)$are AH-nilpotent.

(g) If $N_n(H), N_n(K)$ are AH-solvable, then $N_n(H).N_n(K), N_n(G) \times N_n(G), N_n(G) \cap N_n(G)$ are AH-solvable.

(h) If $N_n(H), N_n(K)$ are AH- meta abelian, then $N_n(H) \cap N_n(K), N_n(G) \times N_n(G)$are AH-meta abelian.

Proof:

It is well known from the classical group theoretical properties that $H_i \cap K_i$ is a subgroup of G, $H_i \times K_i$ is a subgroup of $G \times G$, and $H_i.K_i$ is a subgroup of G under the assumption of normality of H_i, K_i, thus (a),(b),(c) are true.

Also, the direct product and the intersection of any two abelian, nilpotent, or solvable subgroups is the same, thus

(d),(f),(g) are true.

(h), (e) hold by the same argument.

Theorem 3.9:

Let G,K be any two groups, $N_n(G), N_n(K)$ be their corresponding n-refined neutrosophic groups,

$N_n(H) = (H_0, H_1I_1, \dots, H_nI_n)$; $H_i \leq G$ be an AH-subgroup, $f_i: G \to K; 0 \leq i \leq n$ be a classical homomorphism for all $i, f: N_n(G) \to N_n(K); f(a_0, a_1I_1, .., a_nI_n) = (f_0(a_0), f_1(a_1)I_1, \dots, f_n(a_n)I_n)$ be an AH-homomorphism, we have:

(a) If $N_n(H)$ is AH-cyclic/AH-abelian, then $f(N_n(H))$ is AH-cyclic/AH-abelian.

(b) If $N_n(H)$ is AH-nilpotent/AH-solvable, then $f(N_n(H))$ is AH-nilpotent/AH-solvable.

(c) If $N_n(H)$ is AH-meta abelian, then $f(N_n(H))$ is AH-meta abelian.

(d) If $N_n(H)$ is AH-normal, then $f(N_n(H))$ is AH-normal.

(e) AH-Ker(f) is an AH-normal subgroup of $N_n(G)$.

(f) AH-Im(f) is an AHS-subgroup of $N_n(K)$.

Proof:

(a),(b),(c),(d) It is well known that the homomorphic image of cyclic, abelian, nilpotent, normal, or meta abelian subgroup is the same, so the proof is complete.

(e) Since $Ker(f_i)$ is a normal subgroup of G for all i, $AH - Ker(f) = (Ker(f_0), Ker(f_1)I_1, .., Ker(f_n)I_n)$ as an AH-normal subgroup.

(f) The proof is similar to (e).

DOI: 10.5281/zenodo.3929792

Theorem 3.10:

Let G be any finite group, $N_n(G)$ be its corresponding n-refined neutrosophic group,

$N_n(H) = (H_0, H_1 I_1, \dots, H_n I_n)$ be any AH-subgroup. Then

(a) $O(N_n(H)) = O(H_0) \times O(H_1) \times .. \times O(H_n)$.

(b) Lagrange's theorem is true for AH-subgroups, i.e $O(N_n(H))$ divides $O(N_n(G))$.

Proof:

(a) Since $N_n(H) = (H_0, H_1 I_1, \dots, H_n I_n) = (h_0, h_1 I_1, .., h_n I_n);\ h_i \in H_i\}$, we get $O(N_n(H)) = O(H_0) \times O(H_1) \times .. \times O(H_n)$.

(b) Since $O(H_i)$ is a divisor of $O(G) = m$, then $O(N_n(H)) = O(H_0) \times O(H_1) \times .. \times O(H_n)$ is a divisor of

$O(N_n(G)) = m^{n+1}$. See [1].

Definition 3.11:

Let $(G, +)$ be any additive abelian group, $N_n(G)$ be its corresponding n-refined neutrosophic group,

$AH - End(N_n(G))$ is defined to be the set of all AH-homomorphisms between $N_n(G)$ and itself.

$AHS - End(N_n(G))$ is defined to be the set of all AHS-homomorphisms between $N_n(G)$ and itself.

Definition 3.12:

Let $(G, +)$ be any additive abelian group, $N_n(G)$ be its corresponding n-refined neutrosophic group.

We define operations on $AH - End(N_n(G))$ as follows:

Let $f: N_n(G) \to N_n(G); f(a_0, a_1 I_1, .., a_n I_n) = (f_0(a_0), f_1(a_1) I_1, \dots, f_n(a_n) I_n)$,

$g: N_n(G) \to N_n(G); g(a_0, a_1 I_1, .., a_n I_n) = (g_0(a_0), g_1(a_1) I_1, \dots, g_n(a_n) I_n)$ be any two AH-endomorphisms,

Addition is defined as $(f + g)(a_0, a_1 I_1, .., a_n I_n) = ([f_0 + g_0](a_0), [f_1 + g_1](a_1) I_1, \dots, [f_n + g_n](a_n) I_n)$.

Multiplication is defined as $(fog)(a_0, a_1 I_1, .., a_n I_n) = ([f_0 o g_0](a_0), [f_1 o g_1](a_1) I_1, \dots, [f_n o g_n](a_n) I_n)$.

It is easy to see that addition and multiplication are well defined.

$f + g, fog$ are AH-endomorphisms, since $f_i + g_i, f_i o g_i$ are two classical endomorphisms for all i.

This means that $AH - End(N_n(G))$ is closed under addition and multiplication.

Theorem 3.13:

Let $(G, +)$ be any additive abelian group, $N_n(G)$ be its corresponding n-refined neutrosophic group. Then

$(AH - End(N_n(G)), +, o)$ is a ring and it is isomorphic to $End(G, +) \times End(G, +) \times .. \times End(G, +)$ (n+1 times).

Proof:

DOI: 10.5281/zenodo.3929792

Simply, we find that $(AH - End(N_n(G)), +)$ is an abelian group.

Also, multiplication is associative and distributive with respect to addition, since it is associative and distributive for each component i. Thus $(AH - End(N_n(G)), +, o)$ has a structure of ring.

The ring isomorphism between $(AH - End(N_n(G)), +, o)$ and

$End(G, +) \times End(G, +) \times .. \times End(G, +)$ (n+1 times) can be defined as follows:

$\varphi: AH - End(N_n(G)) \to End(G, +) \times End(G, +) \times .. \times End(G, +)$;

$\varphi(f_0, f_1 I_1, .., f_n I_n) = (f_0, f_1, .., f_n)$.

Definition 3.14:

Let G be any group, $N_n(G)$ be its corresponding n-refined neutrosophic group,

$N_n(H) = (H_0, H_1 I_1, \dots, H_n I_n)$; $H_i \leq G$ be an AH-subgroup of $N_n(G)$. We call $N_n(H)$an AH-p subgroup if H_i is a p-group for all i.

Clearly, if G is a finite group, then $O(N_n(H))$ is a prime power according to Theorem 3.10.

Definition 3.15:

Let G be any group, $N_n(G)$ be its corresponding n-refined neutrosophic group,

$N_n(H) = (H_0, H_1 I_1, \dots, H_n I_n)$; $H_i \leq G$ be an AH-subgroup of $N_n(G)$. The AH-derived subgroup of H can be defined as $[N_n(H)]' = (H'_0, H'_1 I_1, \dots, H'_n I_n)$; $H'_i =< x^{-1} y^{-1} xy; \forall x, y \in H_i >$.

Theorem 3.16:

Let G be any group, $N_n(G)$ be its corresponding n-refined neutrosophic group,

If $N_n(H) = (H_0, H_1 I_1, \dots, H_n I_n)$; $H_i \leq G$ be any AH-subgroup of $N_n(G)$. Then its AH-derived subgroup is trivial if and only if it is an AH-abelian subgroup.

Proof:

Suppose that $N_n(H) = (H_0, H_1 I_1, \dots, H_n I_n)$ is AH-abelian, hence H_i is abelian for all i. This implies that H'_i is trivial, thus $[N_n(H)]' = (H'_0, H'_1 I_1, \dots, H'_n I_n)$ is trivial.

Conversrly, If $[N_n(H)]' = (H'_0, H'_1 I_1, \dots, H'_n I_n)$ is trivial, we find that H'_i is trivial, thus H_i is abelian subgroup of G, which means that $N_n(H) = (H_0, H_1 I_1, \dots, H_n I_n)$ is AH-abelian.

Remark 3.17:

There is a well known criteria for solvability of any classical group G. This criteria suggests that any group G is solvable if and only if the normal series formed from derived subgroups $(G', G'', \dots)$ reaches to the trivial subgroup {e} after n-steps.

We can use this argument to determine if an AH-subgroup is AH-solvable or not by computing AH-derivatives

$([N_n(H)]', [N_n(H)]'', ..)$. If the previous series reaches to the trivial subgroup of $N_n(G)$, then $N_n(H)$ is an AH-solvable subgroup.

Also, any group G is meta abelian if and only if its derivative is an abelian subgroup. This idea can be generalized into AH-subgroups as follows:

Any AH-subgroup $N_n(H)$ is AH-meta abelian if and only if its AH-derivative subgroup is an AH-abelian.

Remark 3.18

By using the same argument in Remark 3.17, we can define the AH-cenral series to study the AH-nilpotency of and AH-subgroup.

5. Conclusion

In this article we have defined the concept of AH-homomorphism in an n-refined neutrosophic group for the first time. Also, we have introduced some corresponding notions such as AH-endomorphism, AH-solvability, AH-nilpotency, and other related concepts. Many examples and theorems were constructed to clarify the validity of these concepts.

Funding: “This research received no external funding”

Conflicts of Interest: “The authors declare no conflict of interest.”

References

[1] Abobala, M., " n-Refined Neutrosophic Groups I", International Journal of Neutrosophic Science, submitted 2019.

[2] Agboola, A.A.A,. Akwu, A.D,. and Oyebo, Y.T,. "Neutrosophic Groups and Subgroups", International .J .Math.Combin, Vol. 3, pp. 1-9, 2012.

[3] Agboola, A.A.A., Akinola, A.D., and Oyebola, O.Y.," NeutrosophicRings I ", International J.Mathcombin, Vol. 4, pp. 1-14, 2011.

[4] Agboola, A.A.A., "On Refined Neutrosophic Algebraic Structures", Neutrosophic Sets and Systems, Vol. 10, pp. 99-102, 2015.

[5] Haushi, M., " Algebraic Structures 1", Tishreen University Press, pp. 102-209, 2005.

[6] Kandasamy, V.W.B., and Smarandache, F., "Some Neutrosophic Algebraic Structures and Neutrosophic N-Algebraic Structures", Hexis, Phonex, Arizona 2006.

[7] Smarandache, F., "Symbolic Neutrosophic Theory", EuropaNova asbl, Bruxelles, 2015.

[8] Smarandache, F., "n-Valued Refined Neutrosophic Logic and Its Applications in Physics", Progress in Physics, Vol. 4, pp.143-146, 2013.

[9] Smarandache, F., "Refined Literal Indeterminacy and the Multiplication Law of Sub-Indeterminacies", Neutrosophic Sets ans Systems, Vol. 9, 2015.

[10] Deli, I., " Refined Neutrosophic Sets and Refined Neutrosophic Soft Sets: Theory and Applications", Hand Book of Research on generalized and Hybrid Set Structures and Applications for Soft Computing, 2016.

[11] Rotman, J., "The Theory of Groups", University of Illinois, Urbana, pp. 120-180, 1971.

DOI: 10.5281/zenodo.3929792

Generalized Weighted Exponential Similarity Measures of Single Valued Neutrosophic Sets

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Abstract

A single valued neutrsophic set is one of the most successful extensions of the classical set, fuzzy set, intuitionistic fuzzy set, Pythagorean fuzzy set and q-rung orthopair fuzzy set due to the fact that it can handle uncertain data in more wider way. In this paper, we introduce some new generalized weighted similarity measures based on the exponential functions defined on truth-membership function, indeterminacy membership function and falsity membership function of a single valued neutrosophic set to study the independent influences of the truth-membership function, indeterminacy membership function and falsity membership function. The salient features of these proposed similarity measures are studied in detail. Based on the proposed similarity measures, we propose a multi attribute decision making method. To show the feasibility and effectiveness of the proposed method, an investment decision making problem is demonstrated.

Keywords: Single valued neutrosophic set, weighted exponential similarity measures, decision making.

1. Introduction

In our daily life, we come across various types of multi-attribute decision making problems with non-crisp/ uncertain data. Fuzzy set theory is one such extremely useful tool that helps us to deal with non-crisp data. In 1965, Lotfi A. Zadeh **[1]** first published the famous research paper on fuzzy sets that originated due to mainly the inclusion of vague human assessments in computing problems and it can deal with uncertainty, vagueness, partially trueness, impreciseness, Sharpless boundaries etc. Basically, the theory of fuzzy set is founded on the concept of partial belongings of an element in a set in order to process inexact information. Later on, fuzzy sets have been generalized to intuitionistic fuzzy sets **[2]** by adding a non-membership function by Atanassov in 1986 in order to deal with problems that possess incomplete information. In case of fuzzy sets or intuitionistic fuzzy sets, it is known that the membership (or non-membership) value of an element in a set takes a unique value in the closed interval [0,1]. However, the application range of intuitionistic fuzzy set is narrow in the sense that it has the constraint that sum of membership degree and non-membership degree of an element cannot exceed '1'. But, in complex decision-making problems, decision makers/experts may choose the preferences in such a way that the above condition gets violated. For instance, if an expert gives his preference with membership degree 0.8 and non-membership degree 0.7, then clearly their sum is 1.5, which is greater than 1. Therefore, intuitionistic fuzzy sets are not able to deal with this situation. To solve this problem, Yager **[3, 4]** introduced the non-standard fuzzy set named as Pythagorean fuzzy sets with membership degree ζ and non-membership degree ϑ with the condition $\zeta^2 + \vartheta^2 \leq 1$. Obviously, the Pythagorean fuzzy sets accommodate more uncertainties than the intuitionistic fuzzy sets. Yager **[5]** defined q-rung

DOI: 10.5281/zenodo.3929846
Received: March 28, 2019 Revised: May 24, 2019 Accepted: octobre 09, 2019

orthopair fuzzy sets (*q-ROFSs*) by enlarging the scope of Pythagorean fuzzy sets. The q-rung orthopair fuzzy sets allows the result of the qth power of the membership grade plus the qth power of the non-membership grade to be limited in interval [0,1]. If q=1, the q-rung orthopair fuzzy set transforms into the intuitionistic fuzzy set; if q=2, the q-rung orthopair fuzzy set transforms into the Pythagorean fuzzy set, which means that the q-rung orthopair fuzzy sets are extensions of intuitionistic fuzzy sets and Pythagorean fuzzy sets.

In 1999, Smarandache **[6]** introduced the notion neutrsophic set as a generalization of the classical set, fuzzy set, intuitionistic fuzzy set, Pythagorean fuzzy set and q-rung orthopair fuzzy set. The characterization of this neutrosophic set is explicitly done by truth-membership function, indeterminacy membership function and falsity membership function. The concept of single valued neutrosophic set was developed by Wang et al. **[7]** as an extension of fuzzy sets, Pythagorean fuzzy sets, q-rung orthopair fuzzy sets, intuitionistic fuzzy sets, single valued spherical neutrosophic sets **[8]**, n-hyperspherical neutrosophic sets **[8]**. The possible applications of neutrosophic sets and single valued neutrosophic sets on image segmentation have been studied in Gou and Cheng [9], Gou and Sensur [10]. Also, we find their probable infliction on clustering analysis in Karaaslan [11] and on medical diagnosis problems in Ansari et al. [12] respectively. Furthermore, the subject of the neutrosophic set theory has been practiced in Wang et al. [13], Gou et al. [14], Ye [15], Sun et al. [16], Ye [17, 18, 19] and Abdel Basset et al. [20, 21]. Some recent studies on this area can be found in [22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34].

Similarity measure plays a significant role for measuring the uncertain information. The fuzzy similarity measure is a measure that depicts the closeness among fuzzy sets. Many researchers have conducted extensive studies on similarity measures between fuzzy sets. Zwick et al. [35] reviewed and compared several similarity measures between fuzzy sets based on both geometric and set-theoretic ways. Pappis and Karacapilidis [36] introduced three similarity measures between fuzzy sets. Some more works on similarity measures in fuzzy environment can be found in [37], [38], [39], [40], [41]. Apart from these, some similarity measures in intuitionistic fuzzy environment are summarized in **[42, 43, 44, 45, 46, 47, 48]**. Similarity measures of single valued neutrosophic sets were introduced by Majumdar and Samanta **[49]**. Some authors **[50, 51, 52]** studied the concept of similarity measure between the two single valued neutrosophic sets which are useful to identify whether two sets are identical or atleast to what degree they are identical.

In case of the existing similarity measures **[49, 50, 51, 52]** of single valued neutrosophic sets, the independent influences of the truth-membership function, indeterminacy membership function and falsity membership function are completely ignored. To extend the existing similarity measures, in this paper, we introduce some new generalized weighted similarity measures based on the exponential functions defined on truth-membership function, indeterminacy membership function and falsity membership function. We call them "Generalized weighted exponential similarity measures" of single valued neutrosophic sets.

The rest of the paper is arranged as follows:
Some relevant definitions and results are given in Section 2. In Section 3, different types of generalized weighted exponential similarity measures between two single valued neutrosophic sets are introduced. The salient features of these proposed similarity measures are studied in detail. In Section 4, we propose a multi attribute decision making method. To show the feasibility and effectiveness of the proposed method, an investment decision making problem is considered. Section 5 is devoted to comparative study. Section 6 concludes the paper.

2. Preliminaries

In this section, first we recall some basic notions that are relevant to our study.

2.1 Definition: **[7]** A single-valued neutrosophic set (SVNS) ς on the universe set U is given by

$$\varsigma = \{< x, \xi(x), \vartheta(x), \eta(x) >: x \in U\}$$

where the functions $\xi, \vartheta, \eta : U \to [0,1]$ satisfy the condition $0 \le \xi(x) + \vartheta(x) + \eta(x) \le 3$ for every $x \in U$. The functions $\xi(x), \vartheta(x), \eta(x)$ define the degree of truth-membership, indeterminacy-membership and falsity-membership, respectively of $x \in U$.

2.2 Definition: **[7]** Suppose ς and ς' be two single-valued neutrosophic sets on U and are given by

$\varsigma = \{< x, \xi(x), \vartheta(x), \eta(x) >: x \in U\}$ and $\varsigma' = \{< x, \xi'(x), \vartheta'(x), \eta'(x) >: x \in U\}$. Then

(i) $\varsigma \subseteq \varsigma'$ if and only if $\xi(x) \le \xi'(x), \vartheta(x) \ge \vartheta'(x), \eta(x) \ge \eta'(x)\ \forall x \in U$.

(ii) $\varsigma^c = \{< x, \eta(x), 1-\vartheta(x), \xi(x) >: x \in U\}$

(iii) $\varsigma \cup \varsigma' = \{< x, \max(\xi(x), \xi'(x)), \min(\vartheta(x), \vartheta'(x)), \min(\eta(x), \eta'(x)) >: x \in U\}$.

(iv) $\varsigma \cap \varsigma' = \{< x, \min(\xi(x), \xi'(x)), \max(\vartheta(x), \vartheta'(x)), \max(\eta(x), \eta'(x)) >: x \in U\}$.

2.3 Definition: [49] Let $SVNS^U$ be the collection of all single-valued neutrosophic sets on U. Suppose ς, $\varsigma' \in SVNS^U$ and are given by: $\varsigma = \{< x, \xi(x), \vartheta(x), \eta(x) >: x \in U\}$ and $\varsigma' = \{< x, \xi'(x), \vartheta'(x), \eta'(x) >: x \in U\}$. Then, a similarity measure between ς and ς' is a function defined as $S: SVNS^U \to [0,1]$ which satisfies the following properties:

(I) $0 \le S(\varsigma, \varsigma') \le 1$

(II) $S(\varsigma, \varsigma') = S(\varsigma', \varsigma)$

(III) $S(\varsigma, \varsigma') = 1$ if and only if $\varsigma = \varsigma'$

(IV) $\varsigma \subseteq \varsigma' \subseteq \varsigma'' \Rightarrow S(\varsigma, \varsigma'') \le \min\{S(\varsigma, \varsigma'), S(\varsigma', \varsigma'')\}$

2.3 Definition: [49] Let $SVNS^U$ be the collection of all single-valued neutrosophic sets on U. Suppose ς, $\varsigma' \in SVNS^U$ and are given by: $\varsigma = \{< x, \xi(x), \vartheta(x), \eta(x) >: x \in U\}$ and $\varsigma' = \{< x, \xi'(x), \vartheta'(x), \eta'(x) >: x \in U\}$. Then, a weighted similarity measure between ς and ς' is defined as:

$$S(\varsigma, \varsigma') = \frac{\sum_x \omega_x \left(\xi(x)\xi'(x) + \vartheta(x)\vartheta'(x) + \eta(x)\eta'(x)\right)^2}{\sum_x \omega_x \left\{\left((\xi(x))^2 + (\vartheta(x))^2 + (\eta(x))^2\right) \times \left((\xi'(x))^2 + (\vartheta'(x))^2 + (\eta'(x))^2\right)\right\}} \quad (x \in U)$$

3. **Exponential similarity measures of *SVNSs*:**

This sections presents various types of generalized weighted exponential similarity measures of *SVNSs*. The basic properties of these newly defined similarity measures are discussed.

3.1 Definition: Let $\Delta_1 = \{< x, \mu_{\Delta_1}(x), \gamma_{\Delta_1}(x), \delta_{\Delta_1}(x) >: x \in U\}$ and $\Delta_2 = \{< x, \mu_{\Delta_2}(x), \gamma_{\Delta_2}(x), \delta_{\Delta_2}(x) > : x \in U\}$ be two *SVNSs* over U. For $k \ge 1$ and $x \in U$, let us define three exponential functions:

$$S_x^\mu(\Delta_1, \Delta_2) = e^{-\left|\mu^k{}_{\Delta_1}(x) - \mu^k{}_{\Delta_2}(x)\right|},\ S_x^\gamma(\Delta_1, \Delta_2) = e^{-\left|\gamma^k{}_{\Delta_1}(x) - \gamma^k{}_{\Delta_2}(x)\right|},\ S_x^\delta(\Delta_1, \Delta_2) = e^{-\left|\delta^k{}_{\Delta_1}(x) - \delta^k{}_{\Delta_2}(x)\right|}.$$

3.2 Theorem: Let $\Delta_1 = \{< x, \mu_{\Delta_1}(x), \gamma_{\Delta_1}(x), \delta_{\Delta_1}(x) >: x \in U\}$ and $\Delta_2 = \{< x, \mu_{\Delta_2}(x), \gamma_{\Delta_2}(x), \delta_{\Delta_2}(x) > : x \in U\}$ be two *SVNSs* over U. Then

(a) $0 \le S_x^\mu(\Delta_1, \Delta_2), S_x^\gamma(\Delta_1, \Delta_2), S_x^\delta(\Delta_1, \Delta_2) \le 1$

(b) $S_x^\mu(\Delta_1, \Delta_2) = S_x^\mu(\Delta_2, \Delta_1), S_x^\gamma(\Delta_1, \Delta_2) = S_x^\gamma(\Delta_2, \Delta_1)$ and $S_x^\delta(\Delta_1, \Delta_2) = S_x^\delta(\Delta_2, \Delta_1)$,

(c) $S_x^\mu(\Delta_1, \Delta_2) = S_x^\gamma(\Delta_1, \Delta_2) = S_x^\delta(\Delta_1, \Delta_2) = 1$ if and only if $\Delta_1 = \Delta_2$

(d) If $\Delta_1 \subseteq \Delta_2 \subseteq \Delta_3$, then $S_x^\mu(\Delta_1, \Delta_3) \le \min\{S_x^\mu(\Delta_1, \Delta_2), S_x^\mu(\Delta_2, \Delta_3)\}$, $S_x^\gamma(\Delta_1, \Delta_3) \le \min\{S_x^\gamma(\Delta_1, \Delta_2), S_x^\gamma(\Delta_2, \Delta_3)\}$, $S_x^\delta(\Delta_1, \Delta_3) \le \min\{S_x^\delta(\Delta_1, \Delta_2), S_x^\delta(\Delta_2, \Delta_3)\}$.

Proof: (a)- (c) straight forward.

(d) As $\Delta_1 \subseteq \Delta_2 \subseteq \Delta_3$, we have, $0 \le \mu_{\Delta_1}(x) \le \mu_{\Delta_2}(x) \le \mu_{\Delta_3}(x) \le 1, 1 \ge \gamma_{\Delta_1}(x) \ge \gamma_{\Delta_2}(x) \ge \gamma_{\Delta_3}(x) \ge 0, 1 \ge \delta_{\Delta_1}(x)$

DOI: 10.5281/zenodo.3929846

$\geq \delta_{\Delta_2}(x) \geq \delta_{\Delta_3}(x) \geq 0.$ This gives,

$$0 \leq \mu^k{}_{\Delta_1}(x) \leq \mu^k{}_{\Delta_2}(x) \leq \mu^k{}_{\Delta_3}(x) \leq 1, 1 \geq \gamma^k{}_{\Delta_1}(x) \geq \gamma^k{}_{\Delta_2}(x) \geq \gamma^k{}_{\Delta_3}(x) \geq 0, 1 \geq \delta^k{}_{\Delta_1}(x) \geq \delta^k{}_{\Delta_2}(x), \geq \delta^k{}_{\Delta_3}(x) \geq 0.$$

Now $\left|\mu^k{}_{\Delta_1}(x) - \mu^k{}_{\Delta_3}(x)\right| = \left|\mu^k{}_{\Delta_1}(x) - \mu^k{}_{\Delta_2}(x) + \mu^k{}_{\Delta_2}(x) - \mu^k{}_{\Delta_3}(x)\right|$

$$\leq \left|\mu^k{}_{\Delta_1}(x) - \mu^k{}_{\Delta_2}(x)\right| + \left|\mu^k{}_{\Delta_2}(x) - \mu^k{}_{\Delta_3}(x)\right|$$

$$\Rightarrow -\left|\mu^k{}_{\Delta_1}(x) - \mu^k{}_{\Delta_3}(x)\right| \geq -\left|\mu^k{}_{\Delta_1}(x) - \mu^k{}_{\Delta_2}(x)\right| - \left|\mu^k{}_{\Delta_2}(x) - \mu^k{}_{\Delta_3}(x)\right|$$

$$\Rightarrow e^{-\left|\mu^k{}_{\Delta_1}(x) - \mu^k{}_{\Delta_3}(x)\right|} \leq e^{-\left|\mu^k{}_{\Delta_1}(x) - \mu^k{}_{\Delta_2}(x)\right|} \times e^{-\left|\mu^k{}_{\Delta_2}(x) - \mu^k{}_{\Delta_3}(x)\right|}$$

$$\Rightarrow S_x^\mu(\Delta_1, \Delta_3) \leq S_x^\mu(\Delta_1, \Delta_2) \times S_x^\mu(\Delta_2, \Delta_3) \leq \min\{S_x^\mu(\Delta_1, \Delta_2), S_x^\mu(\Delta_2, \Delta_3)\}$$

Similarly, $S_x^\gamma(\Delta_1, \Delta_3) \leq \min\ \{S_x^\gamma(\Delta_1, \Delta_2), S_x^\gamma(\Delta_2, \Delta_3)\}$, $S_x^\delta(\Delta_1, \Delta_3) \leq \min\ \{S_x^\delta(\Delta_1, \Delta_2), S_x^\delta(\Delta_2, \Delta_3)\}$.

Next, we define the generalized weighted exponential similarities measures for *SVNSs* using the exponential functions defined in definition 3.1.

3.3 Definition: Let $\Delta_1 = \{< x, \mu_{\Delta_1}(x), \gamma_{\Delta_1}(x), \delta_{\Delta_1}(x) >: x \in U\}$ and $\Delta_2 = \{< x, \mu_{\Delta_2}(x), \gamma_{\Delta_2}(x), \delta_{\Delta_2}(x) > : x \in U\}$ be two *SVNSs* over U. Also let $\omega_x > 0$ denotes the weight of the element $x \in U$ such that $\sum_x \omega_x = 1$ .Then we define the generalized weighted exponential similarity measure between the *SVNSs* Δ_1 and Δ_2 as:

$$S_\omega^k(\Delta_1, \Delta_2) = \sum_x \omega_x \times S_x^\mu(\Delta_1, \Delta_2) \times S_x^\gamma(\Delta_1, \Delta_2) \times S_x^\delta(\Delta_1, \Delta_2)$$

3.4 Theorem: Let $\Delta_1 = \{< x, \mu_{\Delta_1}(x), \gamma_{\Delta_1}(x), \delta_{\Delta_1}(x) >: x \in U\}$ and $\Delta_2 = \{< x, \mu_{\Delta_2}(x), \gamma_{\Delta_2}(x), \delta_{\Delta_2}(x) > : x \in U\}$ be two *SVNSs* over U. Then

(a) $0 \leq S_\omega^k(\Delta_1, \Delta_2) \leq 1$

(b) $S_\omega^k(\Delta_1, \Delta_2) = S_\omega^k(\Delta_2, \Delta_1)$

(c) $S_\omega^k(\Delta_1, \Delta_2) = 1$ if and only if $\Delta_1 = \Delta_2$

(d) If $\Delta_1 \subseteq \Delta_2 \subseteq \Delta_3$, then $S_\omega^k(\Delta_1, \Delta_3) \leq \min\ \{S_\omega^k(\Delta_1, \Delta_2), S_\omega^k(\Delta_2, \Delta_3)\}$.

Proof: (a)-(c) straight forward.

(d) For the *SVNSs* $\Delta_1, \Delta_2, \Delta_3$ satisfying $\Delta_1 \subseteq \Delta_2 \subseteq \Delta_3$, we observe from theorem 3.2 that,

$S_x^\mu(\Delta_1, \Delta_3) \leq \min\ \{S_x^\mu(\Delta_1, \Delta_2), S_x^\mu(\Delta_2, \Delta_3)\}, S_x^\gamma(\Delta_1, \Delta_3) \leq \min\ \{S_x^\gamma(\Delta_1, \Delta_2), S_x^\gamma(\Delta_2, \Delta_3)\}$, and $S_x^\delta(\Delta_1, \Delta_3) \leq$ $\min\{S_x^\delta(\Delta_1, \Delta_2), S_x^\delta(\Delta_2, \Delta_3)\}\ \forall x \in U$.

Using these, we get from definition 3.3,

$$S_\omega^k(\Delta_1, \Delta_3)$$

$$\leq \sum_x \omega_x \times \min\ \{S_x^\mu(\Delta_1, \Delta_2), S_x^\mu(\Delta_2, \Delta_3)\} \times \min\ \{S_x^\gamma(\Delta_1, \Delta_2), S_x^\gamma(\Delta_2, \Delta_3)\} \times \min\{S_x^\delta(\Delta_1, \Delta_2), S_x^\delta(\Delta_2, \Delta_3)\}$$

$$\leq \min\left\{\sum_x \omega_x \times S_x^\mu(\Delta_1, \Delta_2) \times S_x^\gamma(\Delta_1, \Delta_2) \times S_x^\delta(\Delta_1, \Delta_2), \sum_x \omega_x \times S_x^\mu(\Delta_2, \Delta_3) \times S_x^\gamma(\Delta_2, \Delta_3) \times S_x^\delta(\Delta_2, \Delta_3)\right\}$$

$$= \min\left\{S_\omega^k(\Delta_1, \Delta_2), S_\omega^k(\Delta_2, \Delta_3)\right\}$$

Next we define the generalized weighted average exponential similarity measure of *SVNSs*.

3.5 Definition: Let $\Delta_1 = \{< x, \mu_{\Delta_1}(x), \gamma_{\Delta_1}(x), \delta_{\Delta_1}(x) >: x \in U\}$ and $\Delta_2 = \{< x, \mu_{\Delta_2}(x), \gamma_{\Delta_2}(x), \delta_{\Delta_2}(x) >$

DOI: 10.5281/zenodo.3929846

$: x \in U\}$ be two *SVNSs* over U. Then the generalized weighted average exponential similarity measure between Δ_1 and Δ_2 is defined as:

$$ {}_A S_\omega^k\left(\Delta_1,\Delta_2\right)=\sum_x \omega_x \times \left\{\frac{S_x^\mu(\Delta_1,\Delta_2)+S_x^\gamma(\Delta_1,\Delta_2)+S_x^\delta(\Delta_1,\Delta_2)}{3}\right\} $$

where $\omega_x > 0$ denotes weight of $x \in U$ such that $\sum_x \omega_x = 1$.

3.6 Theorem: Let $\Delta_1 = \{< x, \mu_{\Delta_1}(x), \gamma_{\Delta_1}(x), \delta_{\Delta_1}(x) >: x \in U\}$ and $\Delta_2 = \{< x, \mu_{\Delta_2}(x), \gamma_{\Delta_2}(x), \delta_{\Delta_2}(x) > : x \in U\}$ be two *SVNSs* over U. Then

(a) $0 \le {}_A S_\omega^k\left(\Delta_1,\Delta_2\right) \le 1$

(b) ${}_A S_\omega^k\left(\Delta_1,\Delta_2\right) = {}_A S_\omega^k\left(\Delta_2,\Delta_1\right)$

(c) ${}_A S_\omega^k\left(\Delta_1,\Delta_2\right) = 1$ if and only if $\Delta_1 = \Delta_2$

(d) If $\Delta_1 \subseteq \Delta_2 \subseteq \Delta_3$, then ${}_A S_\omega^k\left(\Delta_1,\Delta_3\right) \le \min\{{}_A S_\omega^k\left(\Delta_1,\Delta_2\right), {}_A S_\omega^k(\Delta_2,\Delta_3)\}$.

Proof: (a)-(c) straight forward.

(d) For the *SVNSs* $\Delta_1, \Delta_2, \Delta_3$ satisfying $\Delta_1 \subseteq \Delta_2 \subseteq \Delta_3$, we observe from theorem 3.2 that, $S_x^\mu(\Delta_1,\Delta_3) \le \min\{S_x^\mu(\Delta_1,\Delta_2), S_x^\mu(\Delta_2,\Delta_3)\}$, $S_x^\gamma(\Delta_1,\Delta_3) \le \min\{S_x^\gamma(\Delta_1,\Delta_2), S_x^\gamma(\Delta_2,\Delta_3)\}$, and $S_x^\delta(\Delta_1,\Delta_3) \le \min\{S_x^\delta(\Delta_1,\Delta_2), S_x^\delta(\Delta_2,\Delta_3)\}\ \forall x \in U$.

Using these, we get from definition 3.5,

$$
\begin{aligned}
&{}_A S_\omega^k\left(\Delta_1,\Delta_3\right)\\
&=\sum_x \omega_x \times \left\{\frac{S_x^\mu(\Delta_1,\Delta_3)+S_x^\gamma(\Delta_1,\Delta_3)+S_x^\delta(\Delta_1,\Delta_3)}{3}\right\}\\
&\le \frac{1}{3}\left\{\sum_x \omega_x \times \min\{S_x^\mu(\Delta_1,\Delta_2), S_x^\mu(\Delta_2,\Delta_3)\} + \sum_x \omega_x \times \min\{S_x^\gamma(\Delta_1,\Delta_2), S_x^\gamma(\Delta_2,\Delta_3)\} \times\right.\\
&\left.\sum_x \omega_x \times \min\{S_x^\delta(\Delta_1,\Delta_2), S_x^\delta(\Delta_2,\Delta_3)\}\right\}\\
&\le \min\left\{\sum_x \omega_x \times \frac{S_x^\mu(\Delta_1,\Delta_2)+S_x^\gamma(\Delta_1,\Delta_2)+S_x^\delta(\Delta_1,\Delta_2)}{3}, \sum_x \omega_x \times \frac{S_x^\mu(\Delta_2,\Delta_3)+S_x^\gamma(\Delta_2,\Delta_3)+S_x^\delta(\Delta_2,\Delta_3)}{3}\right\}\\
&= \min\left\{{}_A S_\omega^k\left(\Delta_1,\Delta_2\right), {}_A S_\omega^k\left(\Delta_2,\Delta_3\right)\right\}
\end{aligned}
$$

4. **Multi attribute decision making:**

Let $A = \{A_1, A_2, A_3, \ldots\ldots, A_m\}$ be a set of m alternatives and $C = \{C_1, C_2, C_3, \ldots\ldots, C_n\}$ be a sets of n attributes. Suppose ω_j is the weight of the attribute C_j with $\omega_j > 0$ and $\sum_{j=1}^n \omega_j = 1$. These alternatives are evaluated by an expert and evaluation values are presented in terms of SVNSs $\xi_{ij} =< \mu_{ij}, \gamma_{ij}, \delta_{ij} >$ such that $\mu_{ij}, \gamma_{ij}, \delta_{ij} \ge 0$ and $\mu_{ij} + \gamma_{ij} + \delta_{ij} \le 3$ are satisfied for each *i, j*.

To determine the best alternatives, the following steps are followed based on the proposed similarity measures:

Step-1: Determine the weight of each criterion.

The weight vector $\omega_{j,r}\left(r=0,1,2,...\right)$ of criteria C_j is determined by using the formula:

$$\omega_{j,r}=\frac{\left(\beta_j\right)^r}{\sum_{j=1}^{n}\left(\beta_j\right)^r},\ r=0,1,2,3....$$

Where $\beta_j=\beta_{1j}+\beta_{2j}+\beta_{3j}$ in which , $\beta_{1j}=\max_i \mu_{ij},\beta_{2j}=\min_i \gamma_{ij},\beta_{3j}=\min_i \delta_{ij}$ for all $i=1,2,3...,n$ such that $\sum_{j=1}^{n}\omega_{j,r}=1,\text{for}\ r=0,1,2,3.....$

Step-2: Determine the ideal values.

Let $C=C'\cup C''$ where C' denotes the set of all cost criteria and C'' denotes the set of all benefit criteria.

The triplets (0,1,1) and (1,0,0) are considered as ideal values corresponding to cost criteria and benefit criteria respectively.

If $A_I(j)$ represent the ideal value for the criteria C_j, then

$$A_I(j)=\begin{cases}(1,0,0) & \text{if } C_j\in C'' \\ (0,1,1) & \text{if } C_j\in C'\end{cases}\qquad (j=1,2,3,.....,n)$$

Suppose A_I denotes the ideal values for all criteria i.e; $A_I=\{A_I(1),A_I(1),A_I(3),A_I(4),A_I(5),A_I(6)\}$.

Step-3: Calculate the similarity measures using S_ω^k or ${}_AS_\omega^k$ between each alternative and it's ideal values .

Step-4: Based on the values of similarity measures, rank the alternatives using the following rule:

$A_p\prec A_q$ if and only if $S_\omega^k(A_p,A_I)<S_\omega^k(A_q,A_I)$ or ${}_AS_\omega^k(A_p,A_I)<{}_AS_\omega^k(A_q,A_I)$ for $p,q\in\{1,2,....,m\}$ $\left(p\neq q\right)$

An illustrative example:

We consider a investment decision making problem given below adapted from **[53]**.

"There are five possible companies A_i $(i=1,2,3,4,5)$ which are considered as alternatives. To evaluate these alternatives, a person hires an investment expert who evaluates these companies under the set of six criteria, namely –technical ability$\left(C_1\right)$, expected benefit $\left(C_2\right)$,competitive power on the market$\left(C_3\right)$, ability to bear risk $\left(C_4\right)$, management capacity $\left(C_5\right)$ and organizational culture$\left(C_6\right)$".

The expert(s) evaluation result for each alternative based on each criteria is depicted in **Table-1**:

Table-1: Initial evaluation result

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	<0.3,0.4,0.2>	<0.4,0.7,0.6>	<0.1,0.4,0.3>	<0.5,0.5,0.2>	<0.4,0.3,0.5>	<0.6,0.1,0.4>
A_2	<0.6,0.1,0.4>	<0.2,0.4,0.5>	<0.5,0.3,0.4>	<0.7,0.4,0.6>	<0.6,0.3,0.6>	<0.5,0.4,0.2>
A_3	<0.7,0.6,0.3>	<0.5,0.4,0.5>	<0.8,0.6,0.3>	<0.7,0.4,0.6>	<0.8,0.6,0.3>	<0.9,0.5,0.3>
A_4	<0.5,0.5,0.2>	<0.3,0.4,0.2>	<0.4,0.3,0.5>	<0.7,0.3,0.5>	<0.1,0.4,0.3>	<0.5,0.5,0.2>
A_5	<0.1,0.4,0.3>	<0.6,0.6,0.4>	<0.5,0.5,0.2>	0.4,0.3,0.5>	<0.3,0.4,0.2>	<0.1,0.4,0.3>

Then to find the best alternative(s), the following steps are executed:

Step-1: Take r=1.

Then we have,

DOI: 10.5281/zenodo.3929846

$\beta_1 = \beta_{11} + \beta_{21} + \beta_{31} = 0.7 + 0.1 + 0.2 = 1,\qquad \beta_2 = \beta_{12} + \beta_{22} + \beta_{32} = 0.8 + 0.4 + 0.2 = 1.4,$

$\beta_3 = \beta_{13} + \beta_{23} + \beta_{33} = 0.8 + 0.3 + 0.2 = 1.3,\qquad \beta_4 = \beta_{14} + \beta_{24} + \beta_{34} = 0.7 + 0.3 + 0.2 = 1.2,$

$\beta_5 = \beta_{15} + \beta_{25} + \beta_{35} = 0.8 + 0.3 + 0.2 = 1.3,\qquad \beta_6 = \beta_{16} + \beta_{26} + \beta_{36} = 0.9 + 0.1 + 0.2 = 1.2.$

$$\therefore\ \omega_{1,1} = \frac{1}{1+1.4+1.3+1.2+1.3+1.2} = 0.135,\quad \omega_{2,1} = \frac{1.4}{1+1.4+1.3+1.2+1.3+1.2} = 0.189,$$

$$\omega_{3,1} = \frac{1.3}{1+1.4+1.3+1.2+1.3+1.2} = 0.176,\quad \omega_{4,1} = \frac{1.2}{1+1.4+1.3+1.2+1.3+1.2} = 0.162,$$

$$\omega_{5,1} = \frac{1.3}{1+1.4+1.3+1.2+1.3+1.2} = 0.176,\quad \omega_{6,1} = \frac{1.2}{1+1.4+1.3+1.2+1.3+1.2} = 0.162.$$

Step-2: As $C_4 \in C'$ and $C_1, C_2, C_3, C_5, C_6 \in C''$, so the ideal values are given by:

$A_I(1) = (1,0,0), A_I(2) = (1,0,0) >, A_I(3) = (1,0,0), A_I(4) = (0,1,1), A_I(5) = (1,0,0), A_I(6) = (1,0,0).$

Step-3: Using the similarity measure S_ω^k, (for k=2) we get,

$S_\omega^2(A_1, A_I) = 0.2792, S_\omega^2(A_2, A_I) = 0.3172, S_\omega^2(A_3, A_I) = 0.3854, S_\omega^2(A_4, A_I) = 0.2911, S_\omega^2(A_5, A_I) = 0.2916.$

Step-4: Since $S_\omega^2(A_1, A_I) < S_\omega^2(A_4, A_I) < S_\omega^2(A_5, A_I) < S_\omega^2(A_2, A_I) < S_\omega^2(A_3, A_I)$, the best alternative is A_3 i.e; the best company is A_3. However the overall ranking is: $A_1 \prec A_4 \prec A_5 \prec A_2 \prec A_3$.

In another aspect, if we apply the other proposed similarity measure namely, ${}_A S_\omega^k$, then the problem can be solved similarly as above. If we utilize the similarity measure S_ω^k or ${}_A S_\omega^k$ for different values of k, then the final the ranking order of the given alternatives are summarized in **Table-2**. We can conclude from table-2 that although the ranking orders of the alternatives are slightly different; the most desirable alternative is still A_3 in all cases.

Table-2: Ranking of alternatives

Value of k	Similarity measures used	A_1	A_2	A_3	A_4	A_5	Ranking order
		Overall measure values					
1	S_ω^k	0.2332	0.2731	**0.3139**	0.2487	0.2474	$A_1 \prec A_5 \prec A_4 \prec A_2 \prec A_3$
	${}_A S_\omega^k$	0.6186	0.6489	**0.6716**	0.6305	0.6304	$A_1 \prec A_5 \prec A_4 \prec A_2 \prec A_3$
2	S_ω^k	0.2792	0.3172	**0.3854**	0.2911	0.2916	$A_1 \prec A_4 \prec A_5 \prec A_2 \prec A_3$
	${}_A S_\omega^k$	0.6756	0.6985	**0.7239**	0.6889	0.6954	$A_1 \prec A_4 \prec A_5 \prec A_2 \prec A_3$
3	S_ω^k	0.2997	0.3292	**0.4162**	0.3114	0.3151	$A_1 \prec A_4 \prec A_5 \prec A_2 \prec A_3$
	${}_A S_\omega^k$	0.7091	0.7244	**0.7506**	0.7191	0.7277	$A_1 \prec A_4 \prec A_2 \prec A_5 \prec A_3$
4	S_ω^k	0.3105	0.3316	**0.4199**	0.3203	0.3245	$A_1 \prec A_4 \prec A_5 \prec A_2 \prec A_3$
	${}_A S_\omega^k$	0.7275	0.7373	**0.7629**	0.7343	0.7422	$A_1 \prec A_4 \prec A_2 \prec A_5 \prec A_3$
5	S_ω^k	0.3170	0.3314	**0.4131**	0.3245	0.3280	$A_1 \prec A_4 \prec A_5 \prec A_2 \prec A_3$
	${}_A S_\omega^k$	0.7380	0.7441	**0.7681**	0.7424	0.7488	$A_1 \prec A_4 \prec A_2 \prec A_5 \prec A_3$

DOI: 10.5281/zenodo.3929846

5. Comparative study

In pursuance of performance comparison of the weighted exponential similarity measures developed by us with the existing weighted similarity measure **[49]**, a comparative study alongside their corresponding final ranking are summarized in tabular form, numbered by 3. It is very much translucent from table 3 that in spite of appearance of slight difference occur to the respective ranking order of the alternatives, the best i.e. most desirable alternative is absolutely same.

Table-3: Comparative study

		Overall measure values					
Value of *k*	**Similarity measures used**	A_1	A_2	A_3	A_4	A_5	**Ranking order**
K=2	S_{ω}^{k}	0.2792	0.3172	**0.3854**	0.2911	0.2916	$A_1 \prec A_4 \prec A_5 \prec A_2 \prec A_3$
	${}_{A}S_{\omega}^{k}$	0.6756	0.6985	**0.7239**	0.6889	0.6954	$A_1 \prec A_4 \prec A_5 \prec A_2 \prec A_3$
	S_{ω} [49]	0.2843	0.3661	**0.4823**	0.2342	0.2629	$A_4 \prec A_5 \prec A_1 \prec A_2 \prec A_3$

6. Conclusion

In this paper, some new weighted exponential similarity measures between single valued neutrosophic sets have been introduced. The desirable properties of these proposed similarity measures are demonstrated. To show the efficiency of the proposed similarity measures, a multi-attribute decision making method is constructed. The proposed approach is examined on a investment decision making problem. Finally we did a comparative analysis of the proposed approach and get ensured about its best performance.

Funding: This research received no external funding

Conflicts of Interest: The authors declare that they have no conflict of interest.

References

[1] Zadeh, L.A. “Fuzzy Sets”. Information and Control, 8, pp.338-353, 1965.
[2] Atanassov, K.T. “Intuitionistic fuzzy sets”. Fuzzy Sets and Systems , 20 , pp.87–96, 1986.
[3] Yager, R.R. “Pythagorean fuzzy subsets”. Proceedings of joint IFSA World Congress and NAFIPS Annual meeting, 24-28th June, 2013.
[4] Yager, R.R. “Pythagorean membership grades in multi-criteria decision making”. IEEE Transactions on Fuzzy Systems , 22, pp.958-965, 2013.
[5] Yager, R.R. “Generalized orthopair fuzzy sets”. IEEE Transactions on Fuzzy Systems 2016.
[6] Smarandache, F. “A Unifying Field in Logics. Neutrosophy: Neutrosophicprobability, set and logic”.American Research Press, Rehoboth, 1999.
[7] Wang, H.; Smarandache, F.; Zhang ,Y.Q.; Sunderraman, R. “Single valued neutrosophic sets”. Multi-space and Multi-structure , 4 , pp.410–413, 2010.
[8] Smarandache, F. “Neutrosophic set is a generalization of intuitionistic fuzzy set, inconsistent intuitionistic fuzzy set, pythagorean fuzzy set, q-rung orthopair fuzzy set, spherical fuzzy set and n-hyperbolic fuzzy set while neutrosophication is a generalization ofregret theory, grey system theory and three ways decision”. Journal of New Theory, 29, pp.1-35, 2019.
[9] Gou Y., Cheng H.D., “New neutrosophic approach to image segmentation”, Pattern Recognition, 42, pp.587-595, 2009.
[10] Guo, Y.H.; Sensur, A. “A novel image segmentation algorithm based on neutrosophic similarity clustering”. Applied Soft Computing , 25 , pp.391-398, 2014.
[11] Karaaslan, F. “Correlation coefficients of single valued neutrosophic refined soft sets and their applications in clustering analysis”. Neural Computing and Applications , 28, pp.2781-2793,2017.
[12] Ansari, A.Q.; Biswas, R.; Aggarwal, S. “Proposal for applicability of neutosophic set theory in medical AI”. International Journal of Computer Applications 2011, 27(5), pp.5-11, 2011.

[13] Wang, H.; Smarandache, F.; Zhang, Y.Q. "Interval neutrosophic sets and logic: Theory and applications in computing". Hexis, Phoenix, 2005.

[14] Gou, Y.; Cheng, H.D.; Zhang, Y.; Zhao, W. "A new neutrosophic approach to image de-noising". New Mathematics and Natural Computation , 5, pp.653-662, 2009.

[15] Ye, J. "Multi-criteria decision making method using the correlation coefficient under single valued neutrosophic environment". International Journal of General Systems, 42, pp.386-394, 2013.

[16] Sun, H.X.; Yang, H.X.; Wu, J.Z.; Yao, O.Y. "Interval neutrosophic numbers choquet integral operator for multi-criteria decision making". Journal of Intelligent and Fuzzy Systems , 28 , pp.2443-2455, 2015.

[17] Ye, J. "Multiple-attribute decision-making method under a single-valued neutrosophic hesitant fuzzy environment". Journal of Intelligent and Fuzzy Systems, 24(1), pp.23–36, 2015.

[18] Ye, J. "Trapezoidal neutrosophic set and its application to multiple attribute decision-making". Neural Computing and Applications , 26, pp.1157–1166, 2015.

[19] Ye, J. "Some Weighted Aggregation Operators of Trapezoidal Neutrosophic Numbers and Their Multiple Attribute Decision Making Method". Informatica, 28(2), pp.387–402, 2017.

[20] Abdel-Basset, M.; Mohamed, M.; Hussien, A.N.; Sangaiah, A.K. "A novel group decision-making model based on triangular neutrosophic numbers". Soft Computing 22, pp.6629–6643, 2017.

[21] Abdel-Basset, M.; Chang, V.; Gamal, A. "Evaluation of the green supply chain management practices: A novel neutrosophic approach". Computers in Industry, 108, pp.210-220, 2019.

[22] Ferhat, T.; Selcuk, T. "Bezier Curve Modeling for Neutrosophic Data Problem", Neutrosophic Sets and Systems, Vol. 16, pp. 3-5, 2017.

[23] Chen, J.Q.; Ye, J. "Some single-valued neutrosophic Dombi weighted aggregation operators for multiple attribute decision making". Symmetry 2017, 9 (doi:10.3390/sym9060082).

[24] Karaaslan, F. "Gaussian single valued neutrosophic numbers and its applications in multi attribute decision making", Neutrosophic sets and Systems, 22, pp.101-117, 2018.

[25] Smarandache, F. "Neutrosophic Modal Logic", Neutrosophic Sets and Systems, vol. 15, pp. 90-96, 2017.

[26] Liu, P.D.; Chu, Y.C.; Li, Y.W.; Chen, Y.B. "Some generalized neutrosophic number Hamacher aggregation operators and their applications to group decision making". International Journal of Fuzzy Systems, 16(2), pp.242-255, 2014.

[27] Liu, P.D.; Wang, Y.M. "Multiple attribute decision making method based on single valued neutrosophic normalized weighted Bonferroni mean". Neural Computing and Applications 2014, 25, pp.2001-2010, 2014

[28] Liu, P.D.; Shi, L.L. "The generalized hybrid weighted average operator based on interval neutrosophic hesitant set and its application to multi-attribute decision making". Neural Computing and Applications, 26, pp.457-471, 2015.

[29] Liu, P. "The aggregation operators based on Archimedean t-Conormand t-Norm for single-valued neutrosophic numbers and their application to decision making". International Journal of Fuzzy S ystems, 18(5),pp. 849–863, 2016.

[30] Iswarya, P.; Bageerathi; K. "A Study on Neutrosophic Frontier and Neutrosophic Semi-frontier in Neutrosophic Topological Spaces", Neutrosophic Sets and Systems, Vol. 1 pp. 6-15, 2017.

[31] Pinaki, M."On New Measures of Uncertainty for Neutrosophic Sets", Neutrosophic Sets and Systems, vol. 17, pp. 50-57, 2017.

[32] Sahidul, I.; Tanmay, K."Neutrosophic Goal Geometric Programming Problem based on Geometric Mean Method and its Application", Neutrosophic Sets and Systems, vol. 19, pp. 80-90, 2018.

[33] Emad Marei, "Single Valued Neutrosophic Soft Approach to Rough Sets", Theory and Application, Neutrosophic Sets and Systems, vol. 20, pp. 76-85, 2018.

[34] Zhao, A.W.; Du, J.G.; Guan,H.J. "Interval valued neutrosophic sets and multi attribute decision making based on generalized weighted aggregation operators". Journal of Intelligent and Fuzzy Systems, 29, pp.2697-2706, 2015.

[35] Zwick, R.; Carlstein, E.; Budescu, D.V. "Measures of similarity among fuzzy concepts: a comparative analysis", International Journal of Approximate Reasoning, 1, pp.221-242, 1987.

[36] Pappis, C.P.; Karacapilidis, N.I. "A comparative assessment of measures of similarity of fuzzy values". Fuzzy Sets and Systems, 56, pp.171-174, 1993.

[37] Liu, X. Entropy, "distance measure and similarity measure of fuzzy sets and their relations", Fuzzy Sets and Systems,52, pp.305-318,1992

[38] Wang, X.; Baets, D.; Kerre, E. "A comparative study of similarity measures". Fuzzy Sets and Systems, 73, pp.259-268, 1995.

DOI: 10.5281/zenodo.3929846

[39] Fan, J.; Xie, W. "Some notes on similarity measure and proximity measure". Fuzzy Sets and Systems 1999, 101,pp.403-412, 1999.
[40] Balopoulos, V.; Hatzimichailidis, A.G.; Papadopoulos, B.K. "Distance and similarity measures for fuzzy operators". Information Sciences,177, pp.2336-2348, 2007.
[41] Bouchon-Meunier, B.; Coletti, G.; Lesot, M; Rifqi, M. "Towards a conscious choice of a fuzzy similarity measure: a qualitative point of view". Lecture Notes in Computer Science, pp.61-78, 2010.
[42] Ye, J. Cosine similarity measures for intuitionistic fuzzy sets and their applications. Mathematical and Computer Modelling, pp.53, 91–97, 2011.
[43] Garg, H; Kumar, K. "An advanced study on the similarity measures of intuitionistic fuzzy sets based on the set pair analysis theory and their application in decision making". Soft Computing, 22, pp.4959–4970, 2018.
[44] Hwang, C.M.; Yang, M.S.; Hung, W.L. "New similarity measures of intuitionistic fuzzy sets based on the jaccard index with its application to clustering". International Journal of Intelligent Systems, 33,pp.1672–1688, 2018.
[45] Chen, S.M.; Chang, C.H. "A novel similarity measure between Atanassov's intuitionistic fuzzy sets based on transformation techniques with applications to pattern recognition". Information Sciences, 291, pp.96–114, 2015.
[46] Hung, W.L.; Yang, M.S. "Similarity measures of intuitionistic fuzzy sets based on hausdorff distance". Pattern Recognition Letters, 25, pp.1603–1611, 2004.
[47] Li, D.F; Cheng, C. "New similarity measures of intuitionistic fuzzy sets and application to pattern recognitions". Pattern Recognition Letters, 23, 221–225, 2002.
[48] Liang, Z.; Shi, P. "Similarity measures on intuitionistic fuzzy sets". Pattern Recognition Letters, 24, pp.2687–2693, 2003.
[49] Majumdar, P.; Samanta, S.K. "On similarity and entropy of neutrosophic sets". Journal of Intelligence and Fuzzy Systems, 26, 1245-1252, 2014.
[50] Broumi, S.; Smarandache, F. "Several similarity measures of neutrosophic sets". Neutrosophic Sets and Systems, 1(1), pp.54-62, 2013.
[51] Broumi, S.; Smarandache, F." Cosine similarity measures of interval valued neutrosophic sets", Neutrosophic Sets and Systems, 5, 15-20, 2013.
[52] Ye, S.; Ye, J. "Dice similarity measure between single valued neutrosophic multi-sets and its application in medical diagnosis". Neutrosophic Sets and Systems, 6, pp.49-54, 2014.
[53] Nguyen X.T.; Nguyen, V.D.; Nguyen, V.H.; Garg, H. "Exponential similarity measures for Pythagorean fuzzy sets and their applications to pattern recognition and decision making process". Complex and Intelligent Systems 5, pp.217-228, 2019.

DOI: 10.5281/zenodo.3929846

A Contribution to Neutrosophic Groups

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Abstract

The objective of this paper is to define some new substructures (AH-substructures) in a neutrosophic group. Also, it deals with some elementary properties of AH-subgroups, AH-normality, AH-homomorphisms, AH-quotients and AH-direct products.

Keywords: Neutrosophic group, AH- subgroup, AH- homomorphism, AH-quotient.

1. Introduction

The fuzzy set and intuitionistic fuzzy set theory were adopted effectively from their initiation to solve optimization problems at vague and uncertain situation in our daily life activities. The intuitionistic fuzzy set theory introduced by Atanassov [5] deals with the degree of belongingness and the degree of non-belongingness of an object to a set simultaneously. Thus it is the more generalization concept than fuzzy set theory which can provide only the degree of belongingness of an object to a set. Both of the theories can only handle incomplete information not indeterminate. To access both incomplete and indeterminate information, Smarandache [8,9] generalized the intuitionistic fuzzy set to neutrosophic set (NS) where each proposition is estimated by three independent parameters namely truth-membership value (T) , indeterminacy membership value (I) and falsity-membership value (F) with $T,I,F \in]^-0,1^+[$ and $^-0 \leq \sup T + \sup I + \sup F \leq 3^+$. Smarandache used to practice the standard or nonstandard subsets of $]^-0,1^+[$ in philosophical ground. So, to incorporate this concept in real life scenario, Wang et al. [10] brought the concept of single valued neutrosophic set which takes the value from real standard subset of [0,1] only.

Also, Smarandache and Vasantha Kandasamy introduced the notion of neutrosophic group in [1]. Neutrosophic rings were introduced in [4]. The neutrosophic group in general does not have a group structure.

DOI: 10.5281/zenodo.3929852

Received: May 26, 2019 Revised: July 22, 2019 Accepted: octobre 14, 2019

In this paper, we shall continue the study of neutrosophic groups by introducing the notion of an AH-subgroup and develop some basic theory, we will use the idea of AH-concepts introduced in [1,2].

Neutrosophic AH-subgroup will be defined as a union of two subgroups $K \cup T$; K is a subgroup of G and T is a subgroup of the pure neutrosophic group GI.

2. Preliminaries

In this following section, we recall some important and useful definitions about neutrosophic groups.

Definition 2.1: [3]

Let (G,*) be a group . Then the neutrosophic group is generated by G and I under * denoted by

N(G)={$< G \cup I >, *$}.

I is called the indeterminate (neutrosophic element) with the property $I^2 = I$.

Definition 2.2: [3]

Let N(G) be a neutrosophic group and H be a neutrosophic subgroup, i.e., (H contains a proper subgroup of G). Then H is a neutrosophic normal subgroup of N(G) if $xH = Hx$ for all $x \in N(G)$.

Definition 2.3: [3]

Let N(G) be a neutrosophic group. Then the center of N(G) is denoted by C(N(G)), and defined

$C(N(G)) = \{x \in N(G); xy = yx \; \forall \, y \in N(G)\}$.

Definition 2.4: [3]

Let N(G) , N(H) be two neutrosophic groups, then $N(G) \times N(H) = \{(g,h); g \in N(G), h \in N(H)\}$.

Definition 2.5: [6]

Let N(G), N(H) be two neutrosophic groups and $\varphi: N(G) \to N(H)$ is called a neutrosophic homomorphism if it is a homomrphism between G , H and $\varphi(I) = I'$.

Where I' is the neutrosophic element of N(H).

If φ is a correspondence one-to-one it is called a neutrosophic isomorphism.

Example 2.6:

Let Z_7 = {0, 1, 2, …, 6} be the group of integers under addition modulo 7. N(G) = {⟨$Z_7 \cup$ I⟩, '+' modulo 7} is a neutrosophic group which is in fact a group. For N(G) = {a + bI ; a, b $\in Z_7$} is a group under '+' modulo 7. Thus this neutrosophic group is also a group.

Theorem 2.7: [3]

Let (G, *) be a group, N(G) = {⟨G $\cup$ I⟩, *} be the neutrosophic group. Then

DOI: 10.5281/zenodo.3929852

N(G) in general is not a group.
N(G) always contains a group.

Example 2.8:

Let N(G) be a neutrosophic group generated by $(Z_7^*, .)$, given by N(G) = {1, 2, 3, 4, 5, 6, I, 2I, 3I, 4I, 5I, 6I} be a neutrosophic group under multiplication modulo 7.

H = {1, 6, I, 6I} is a neutrosophic subgroup of N(G). For x∈ N(G), xH = Hx, so H is normal in N(G).

Definition 2.9: [11]

(a) Let G be any group. It is called meta abelian if it has a abelian derivative subgroup G'.

(b) Let G be a group. It is called nilpotent if it has a central series.

For the concept of central series, see [11].

Remark 2.10: [11]

S_3 is a solvable group, but it is not nilpotent.

D_4 is a meta abelian and nilpotent group.

The intersection, and the direct product of two meta abelian groups is meta abelian.

The intersection, and the direct product of two nilpotent groups is nilpotent.

AH-subgroups were firstly defined in n-refined neutrosophic groups. We recall some definitions.

Definition 2.11: [1]

Let $(G,*)$ be a group, we define the corresponding n-refined neutrosophic group $N_n(G)$ as follows:

$N_n(G) = (< G \cup \{I_1, \dots, I_n\} >, *) = \{(a_0, a_1 I_1, \dots, a_n I_n);\ a_i \in G\}$.

It is easy to see that $N_n(G)$ is closed under $*$, and it is a semigroup but not a group since I_i has no inverse with respect to $*$ in general.

Remark 2.12: [1]

If (G,+) is an additive abelian group, then addition on $N_n(G)$ can be described as follows:

Consider $x = (a_0, a_1 I_1, \dots, a_n I_n), y = (b_0, b_1 I_1, \dots, b_n I_n)$, we have

$x + y = (a_0 + b_0, [a_1 + b_1] I_1, \dots, [a_n + b_n] I_n)$. In this case $(N_n(G), +)$ is a classical abelian group.

The identity element is (0,0,…,0).

It is easy to see that $N_n(G) \cong G \times G \times .. \times G\ (n + 1\ times)$ in the case of abelian additive group G.

Remark 2.13: [1]

If G is a multiplicative group, then group product on $N_n(G)$ can be described as follows:

DOI: 10.5281/zenodo.3929852

Consider $x = (a_0, a_1 I_1, \dots, a_n I_n), y = (b_0, b_1 I_1, \dots, b_n I_n)$, we have

$xy = (t_0, t_1, \dots, t_n);\ t_s = \prod_{i,j=0}^{n}(a_i b_j) I_i I_j\ ;\ I_0 = e_G\ and\ I_i I_j = I_s.$

The identity element is $(e_G, e_G I_1, \dots, e_G I_n)$.

In this case $N_n(G)$ is not isomorphic to the direct product of n+1 copies of G, since it is not a classical group in this case.

The binary operation between the sub-indeterminacies is $I_i . I_j = I_{\min(i,j)}$.

3. Main discussion

Definition 3.1 :

Let $(G, *)$ be any group, the pure neutrosophic group GI is the set $\{x * I : for\ every\ x\ from\ G\}$.

Example 3.2 :

Let $G = (Z_5, +)$, then the corresponding pure neutrosophic group is

$\{0 + I, 1 + I, 2 + I, 3 + I, 4 + I\}$, since (*) is considered as (+) between the elements of G.

Theorem 3.3 :

The set GI has a group structure under the binary operation (xI)(yI)=(xy)I with identity I and $f: G \to GI$;

$f(x) = xI$ is an isomorphism.

Proof :

It is easy to see that GI is a group under the previous operation with identity I.

We have $f(xy) = xyI = (xI)(yI) = f(x)f(y)$, so f is a homomorphism, and f is a one to one correspondence, thus f is an isomorphism clearly.

Remark 3.4 :

For every subgroup H of G, we can find an isomorphic subgroup with form HI of GI.

Definition 3.5 :

Let N(G) be a neutrosophic group and K be a subset of N(G), we say that K is an AH-subgroup if K=$H \cup T$ such that H is a subgroup of G and T is a subgroup of GI.

Example 3.6 :

Let $G = Z_4$, then K={0 , 2 , I} is an AH-subgroup of N(G) because {0, 2 } is a subgroup of G and {I} is a subgroup of the pure neutrosophic group GI under the operation defined in Theorem 3.3.

Definition 3.7 :

Let N(G) be a neutrosophic group, K be an AH-subgroup, we say that K is AHS-subgroup if T≅H.

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Definition 3.8 :

Let N(G) be a neutrosophic group, K be an AH-subgroup we say that K is an AH-normal subgroup if H is normal in G and T is normal in GI.

Definition 3.9 :

(a) Let N(G) be a neutrosophic group and K be an AHS-subgroup, we say that K is an AHS-normal subgroup if it is AH-normal.

(b) Let N(G) be a neutrosophic group, $K = K_1 \cup K_2$, $S = S_1 \cup S_2$ be two AH-subgroups. The intersection is defined as $K \cap S = (K_1 \cap S_1) \cup (K_2 \cap S_2)$.

Theorem 3.10 :

Let N(G) be a neutrosophic group, then

(a) If K, S are two AH-subgroups, $K \cap S$ is AH-subgroup.

(b) If K, S are two AHS-subgroups, $K \cap S$ is AHS-subgroup.

(c) If K, S are two AH-normal subgroups, $K \cap S$ is AH-normal subgroup.

(d) If K, S are two AHS-normal subgroups, $K \cap S$ is AHS-normal subgroup.

Proof :

(a) Suppose that $K = K_1 \cup K_2$, $S = S_1 \cup S_2$, then $K \cap S = (K_1 \cap S_1) \cup (K_2 \cap S_2)$ is an AH-subgroup, because

$K_1 \cap S_1$ is a subgroup of G and $K_2 \cap S_2$ is a subgroup of GI.

(b) The proof holds directly from (a).

(c) If K_1 , S_1 are normal subgroups of G and K_2 , S_2 are normal in GI, then $K_2 \cap S_2$ is normal in GI and $K_1 \cap S_1$ is normal in G, thus $K \cap S = (K_1 \cap S_1) \cup (K_2 \cap S_2)$ is AH-normal subgroup.

(d) It holds from (c).

Definition 3.11 :

Let N(G) be a neutrosophic subgroup with two AH-subgroups K=$K_1 \cup K_2$, S=$S_1 \cup S_2$.

We define KS = $K_1S_1 \cup K_2S_2$.

Theorem 3.12 :

Let N(G) be a neutrosophic subgroup with two AH-normal subgroups K=$K_1 \cup K_2$, S=$S_1 \cup S_2$, then KS is an AH-normal subgroup.

Proof :

Since K_1S_1 is normal subgroup of G and K_2S_2 is normal in GI. The proof is complete.

Definition 3.13:

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Let N(G) be a neutrosophic subgroup with an AH-normal subgroup K=$K_1 \cup K_2$. We define the AH-Quotient N(G)/K= $G/K_1 \cup GI/K_2$.

If K is an AHS-normal subgroup, then N(G)/K is called AHS-Quotient.

The AHS-Quotient N(G)/K must be understood as $G/K_1 \cup (G/K_1)I$ because $K_1 \cong K_2$.

Remark 3.14 :

If K=$K_1 \cup K_2$ and S=$S_1 \cup S_2$ are two AH-subgroups, we say that $K \cong S\ if\ and\ only\ if\ K_1 \cong S_1\ and\ K_2 \cong S_2$.

Theorem 3.15 :

Let N(G) be a neutrosophic group with an AH-normal subgroup K. Then

(a) If N(G) is abelian, then N(G)/K is abelian.

(b) If K is an AHS-normal subgroup, and $xK = yK$ for $x , y \in G$, then $xy^{-1} \in K_1$. Also, if $y = zI \in GI$, then $xz^{-1} \in K_1$.

(c) If G is finite and K is AHS-subgroup, o(K) will divide o(N(G)).

Proof :

(a) It can be proved as the classical case.

(b) Suppose that y$\in G$, then by the proposition we have $xK_1 \cup xIK_2 = yK_1 \cup yIK_2$, so $xK_1 = yK_1$, thus $xy^{-1} \in K_1$.

Now if $y = zI \in GI$, then $xK_1 \cup xIK_2 = zK_1 \cup zIK_2$, hence $xK_1 = zK_1$, thus $xz^{-1} \in K_1$.

(c) It holds directly from Lagrange's theorem.

Example 3.16 :

Let $G = Z_6$ be the group of integers modulo 6 with respect to addition, we have {0, 2, 4, I, 3+I} is an AH-normal subgroup, because {0,2,4} is normal in G and {I , 3+I} is normal in GI. The corresponding AH-quotient is

{ 1+{ 0 , 2, 4 } , {0 , 2, 4 } , {I, 3+I} , (1+I)+{I, 3+I} , (2+I)+{I, 3+I} }.

{0, 3 , I , 3+I } is an AHS-normal subgroup of N(G) and the related AHS-quotient is

{ {0,3} , 1+{0,3} , 2+{0,3} , {I, 3+I} , (1+I)+{I, 3+I} , (2+I)+{I, 3+I} }.

Theorem 3.17 :

Let N(G) be a neutrosophic group, K be an AH-normal subgroup, S be an AH-subgroup of N(G)/K, then there is an AH-subgroup T of N(G) such that S is contained in T as an AH-normal subgroup.

Proof :

Suppose that S is an AH-subgroup of N(G)/K, then $S = S_1 \cup S_2$ such S_1 is a subgroup of $G/K_1$and $S_2\ is\ a\ subgroup$ of GI/K_2, so that $S_1 = T_1/K_1$,$S_2 = T_2/K_2$ where T_1 ,T_2 are two subgroups of G , GI respectively, and $K_1 \leq T_1$,$K_2 \leq T_2$, we put $T = T_1 \cup T_2$, thus we get the proof.

DOI: 10.5281/zenodo.3929852

Definition 3.18 :

Let N(G) , N(H) be two neutrosophic groups with neutrosophic elements I , I' respectively, we define the AH-direct product $N(G) \times N(H)$ as a union

$(G \times H) \cup (G \times HI') \cup (GI \times H) \cup (GI \times HI')$.

For more comprehension of AH-structures we shall introduce the following definition.

Definition 3.19 :

Let N(G) , N(H) be two neutrosophic groups and $f_G: G \to H$ be a homomorphism, we define the corresponding AHS-homomorphism as follows:

$f: N(G) \to N(H); f(x + yI) = f_G(x) + f_G(y)I$.

We define $AH - Ker(f) = Ker(f_G) \cup Ker(f_{GI})$. It is easy to see that $Kerf_G \cong Kerf_{GI}$, so that $AH - Ker(f)$ is an AHS-subgroup of N(G).

We can understand the AH-Kernel as a union, $AH - ker(f) = Kerf_G \cup Kerf_G I$.

Theorem 3.20 :

Let f be an AHS-homomorphism between N(G) and N(H). We have

(a) $AH - Ker(f)$ is an AHS-normal subgroup of N(G).

(b) N(G)/$AH - Ker(f) \cong f_G(G) \cup f_G(G)I$. (The isomorphism here is taken by the concept of AHS-isomorphism).

The quotient in (b) is taken as AH-quotient.

Proof :

(a) Since $Ker(f_G)$ and $Ker(f_{GI})$ are normal in G , GI respectively, $AH - Ker(f)$ is AHS-normal.

(b) We have $G/Kerf_G \cong f_G(G)$ and $GI/Kerf_{GI} \cong f_{GI}(GI)$,

$f(N(G)) = f(G) \cup f(GI) \cong G/Ker(f_G) \cup GI/Ker(f_{GI}) =$ N(G)/$AH - Ker(f)$.

Theorem 3.21 :

Let N(G) be a neutrosophic group and $K = K_1 \cup K_2$, $H = H_1 \cup H_2$ be two neutrosophic AHS-normal subgroups with $H \leq K$. Then

$(N(G)/H)/(K/H) \cong N(G)/K$. (Quotients and isomorphisms are taken as AH-concept).

Proof :

We have $(N(G)/H)/(K/H) = (G \cup GI/H_1 \cup H_2) / (K_1 \cup K_2/H_1 \cup H_2) = G/H_1 \cup (GI/H_2)/(K_1/H_1) \cup (K_2/H_2)$.

So $(N(G)/H)/(K/H) \cong G/H_1/(K_1/H_1) \cup (GI/H_2)/(K_2/H_2) \cong G/K_1 \cup GI/K_2 = N(G)/K$.

Theorem 3.22:

DOI: 10.5281/zenodo.3929852

Let N(G) , N(H) be two neutrosophic groups with two AHS-normal subgroups $K = K_1 \cup K_2$, $S = S_1 \cup S_2$ respectively, Then

$N(G) \times N(H)/K \times S \cong N(G)/K \times N(H)/S$. (Direct products and isomorphisms are taken as AH-concepts).

Proof :

We have $N(G) \times N(H)/K \times S =$
$(G \times H) \cup (G \times HI') \cup (GI \times H) \cup (GI \times HI')/(K_1 \times S_1) \cup (K_1 \times S_2) \cup (K_2 \times S_1) \cup (K_2 \times S_2) =$
$(G \times H/K_1 \times S_1) \cup (G \times HI'/K_1 \times S_2) \cup (GI \times H/K_2 \times S_1) \cup (GI \times HI'/K_2 \times S_2) \cong N(G)/K \times N(H)/S$.

We will construct some examples to clarify the concepts defined previously.

Example 3.23:

Let $G = (Z, +), H = (Z_{12}, +)$ be two groups, $N(G), N(H)$ be their corresponding neutrosophic groups,

$f_G: Z \to Z_{12}; f(x) = x \, mod 12$ be a homomorphism, $Ker(f_G) = 12Z$, $GI = G + I = \{x + I; x \in Z\}$, since the considered operation in G is addition, $HI = H + I = \{x + I; x \in Z_{12}\}$, since the considered operation in H is addition modulo 12.

(a) $S_1 = 3Z, S_2 = 6Z + I = \{6x + I; x \in Z\}$ are two subgroups of G, G+I respectively. $K = S_1 \cup S_2$ is an AH-subgroup of N(G).

(b) $f: N(G) \to N(H); f(x + yI) = f_G(x) + f_G(y)I = (x \, mod 12) + (y \, mod 12)I$ is an AH-homomorphism.

(c) $AH - Ker(f) = Ker(f_G) \cup Ker(f_G)I = 12Z \cup (12Z + I)$ which is an AHS-subgroup.

(d) The AH-quotient
$N(G)/AH - Ker(f) = G/12Z \cup [G/12Z + I] \cong Z_{12} \cup [Z_{12} + I] = H \cup [H + I] = f_G(G) \cup f_G(G)I$.

(d) $f(K) = f_G(S_1) \cup f_G(S_2) = \{0,3,6,9\} \cup (\{0,6\} + I) = \{0,3,6,9, I, 6 + I\}$ which is an AH-subgroup of N(H).

(e) $K, f(K)$ are AH-normal, since commutavity implies normality in abelian groups.

(f) The AH-direct product of N(H) with itself is equal to

$N(H) \times N(H) = (H \times H) \cup (H \times HI) \cup (HI \times H) \cup (HI \times HI)$, according to Definition 3.18.

Example 3.24:

Let $G = Z_{12}$ be the group of integers modulo 6, $H = \{0,2,4,6,8,10\}$, $K = \{0,3,6,9\}, S = \{0,6\}$ are normal subgroups of G.

$M = H \cup KI = \{0,2,4,6,8,10\} \cup (\{0,3,6,9\} + I) = \{0,2,4,6,8,10, I, 3 + I, 6 + I, 9 + I\}$ is an AH-normal subgroup, $N = K \cup SI = \{0,3,6,9\} \cup (\{0,6\} + I) = \{0,3,6,9, I, 6 + I\}$ is another AH-normal subgroup, we clarify Theorem 3.12 as follows:

$MN = HK \cup KSI = \{h + k; h \in H, k \in K\} \cup (\{k + s; k \in K, s \in S\} + I)$=

$\{0,1,2,3,4,5,6,7,8,9,10,11\} \cup \{I, 3 + I, 6 + I, 9 + I\} = \{0,1,2,3,4,5,6,7,8,9,10,11, I, 3 + I, 6 + I, 9 + I\}$, which is an AH-normal subgroup.

DOI: 10.5281/zenodo.3929852

For a future research, we will show some definitions and new concepts related to AH-structures, such as AH-solvability, AH-nilpotency, and AH-cyclicity.

Definition 3.25:

Let $N(G)$ be a neutrosophic group, $N(H) = T \cup S; T \leq G, S \leq GI$ be an AH-subgroup, we say

(a) $N(H)$ is an AH-solvable subgroup if T, S are solvable.

(b) $N(H)$ is an AH-nilpotent subgroup if T, S are nilpotent.

(c) $N(H)$ is an AH-abelian subgroup if T, S are abelian.

(d) $N(H)$ is an AH-cyclic subgroup if T, S are cyclic.

Example 3.26:

Let $G = S_3, T =< (1\ 2) >, S =< (1\ 2\ 3) >$ be two subgroups of G, we have

(a) $SI = \{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} I, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} I, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} I\}$ is a subgroup of GI.

(b) $N(H) = T \cup S = \{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} I, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} I, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} I\}$ is an AH-subgroup of N(G).

(c) N(H) is AH-cyclic and AH-abelian subgroup, since T,S are cyclic and abelian.

(d) N(H) is AH-Nilpotent and AH-solvable, since T,S are nilpotent and solvable.

5. Conclusion

In this article, we have defined the concept of AH-homomorphism in a neutrosophic group for the first time. Also, we have introduced some corresponding notions such as AH-subgroup, AH-normailty, and AH-factors. Many examples and theorems were constructed to clarify the validity of these concepts.

Funding: “This research received no external funding”

Conflicts of Interest: “The authors declare no conflict of interest.”

References

[1] Abobala, M., " n-Refined Neutrosophic Groups I", International Journal of Neutrosophic Science, Vol. 0, No. 1, pp.. 27-34, 2019

[2] Abobala, M., "n-Refined Neutrosophic Groups II", International Journal of Neutrosophic Science, Vol. 0, No. 1, pp. 47-56, 2019

[3] Agboola, A.A.A,. Akwu, A.D,. and Oyebo, Y.T,. "Neutrosophic Groups and Subgroups", International .J .Math.Combin, Vol. 3, pp. 1-9, 2012.

[4] Agboola, A.A.A., Akinola, A.D., and Oyebola, O.Y.," Neutrosophic Rings I ", International J.Math combin, Vol. 4, pp. 1-14, 2011.

DOI: 10.5281/zenodo.3929852

[5] Atanassov, K., "Intuitionistic Fuzzy Sets", Fuzzy Sets and Systems, 20(1), pp.87-96, 1971.

[6] Haushi, M., " Algebraic Structures 1", Tishreen University Press, pp. 102-209, 2005.

[7] Kandasamy, V.W.B., and Smarandache, F., "Some Neutrosophic Algebraic Structures and Neutrosophic N-Algebraic Structures", Hexis, Phonex, Arizona 2006.

[8] Smarandache, F., "Neutrosophy, Neutrosophic Probability, Set and Logic", Amer. Res Press, Rehoboth, USA, http://fs.gallup.unm.edu/eBook-neutrosophics6. pdf (sixth version). 1998.

[9] Smarandache, F., " Neutrosophic Set, A Generalisation of the Intuitionistic Fuzzy Sets", Inter. J. Pure Appl. Math., 24, pp. 287-297, 2005.

[10] Wang, H., Zhang, Y., Sunderraman, R., and Smarandache, F., "Single Valued Neutrosophic Sets, Fuzzy Sets, Rough Sets and Multivalued Operations and Applications", 3(1), pp. 33-39, 2011.

[11] Rotman, J., "The Theory of Groups", University of Illinois, Urbana, pp. 120-180, 1971.

DOI: 10.5281/zenodo.3929852

Star Neutrosophic Fuzzy Topological Space

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Abstract

In this paper, we aim to develop a new type of neutrosophic fuzzy set called the star neutrosophic fuzzy set as a generalization to star neutrosophic crisp set defined in by Salama et al.[8], and study some of its properties. Adedd to, we introduce the notion of star neutrosophic fuzzy topological space as a generalization to some topological consepts as star neutrosophic fuzzy closure, and star neutrosophic fuzzy interior. Finally, we extend the concepts of fuzzy topological space, and intuitionistic fuzzy topological space to the case of star neutrosophic fuzzy sets.

Keywords: Neutrosophic logic, Neutrosophic set; Star neutrosophic fuzzy set, Neutrosophic fuzzy topology, Neutrosophic crisp set

1.Introduction

In 1983 the intuitionistic fuzzy set was introduced by Atanassov et al. [1, 2, 3] as a generalization of fuzzy sets in Zadeh's sense [12], where besides the degree of membership of each element there was considered a degree of non-membership. Smarandache [7, 8, 9], defined the notion of neutrosophic set, which is a generalization of Zadeh's fuzzy sets and Atanassov's intuitionistic fuzzy set. Neutrosophic sets have been investigated by Salama et al. [4-8]. This paper is devoted as a generalization of star neutrosophic crisp set called the fuzzy neutrosophic crisp set. The introduced set is a retraction of any triple structured fuzzy set. Where as, the star set deduced from any neutrosophic crisp set is coincide its corresponding star neutrosophic fuzzy set defined in by Salama et al in [8]. Neutrosophy has laid the foundation for a whole family of new mathematical theories generalizing both their classical and fuzzy counterparts, such as a neutrosophic set theory. The readers can referes the following refrences [13-15] for more informations on the application of neutrosophic theory in divers fields.

2. Preliminaries:

We recollect some relevant basic preliminaries, and in particular, the work of Smarandache in [9-11], Atanassov in [1, 2, 3] and Salama in [4-8]

3. Star Neutrosophic Fuzzy Sets

DOI: 10.5281/zenodo.3930146

Received: May 23, 2019 Revised: July 05, 2019 Accepted: octobre 26, 2019

As a retraction of neutrosophic fuzzy set, we will introducethe star neutrosophic fuzzy set as a generalization of the star neutrosophiccrisp setwas introduced in [5].

3.1 Definition

For a neutrosophic fuzzy set $A = \langle T_A(x),\ I_A(x), F_A(x)\rangle$of the non-empty fixed set X,the star neutrosophic fuzzy set A^*, is defined to be the following triple structure:

$A^* = \langle\, {T_A}^*,\ {I_A}^*,\ {F_A}^*\, \rangle$;where ${T_A}^* = min(T_A, 1-(max(I_A, F_A))$, ${I_A}^* = min(I_A, 1-(max(T_A, F_A))$and

${F_A}^* = min(F_A, 1-(max(T_A, I_A))$;where ${T_A}^*,\ {I_A}^*,\ {F_A}^* : X \to [0,1\,]$.

3.1 Corollary

For any nonempty set X,the star null and the star universe sets (0_N^* and 1_N^*,respectively) are also star neutrosophic fuzzy set.

3.1 Remark

1) All types of 0_N^* and 0_N are conceded.

2) All types of 1_N^* and1_N are conceded.

3. 2 Definition

The complement of a star neutrosophic fuzzy set $A^*(co\, A^*$, for short$)$ may be defined as one of the following two types:

c1:$co\ A^* = \langle\, 1-{T_A}^*, 1-{I_A}^*,\ 1-{F_A}^*\rangle$.

c2:$co\ A^* = \langle {F_A}^*,\ 1-{I_A}^*,\ {T_A}^*\rangle$.

3.3 Definition

The star neutrosophic fuzzy set A^* is said to be a star neutrosophic fuzzy subset of the star neutrosophic fuzzy set B^* ($A^* \subseteq B^*$), and to be defined as one of the following two types:

Type 1:$A^* \subseteq B^* \iff {T_A}^* \le {T_B}^*,\ {I_A}^* \le {I_B}^*$and ${F_A}^* \ge {F_B}^*$.

Type 2:$A^* \subseteq B^* \iff {T_A}^* \le {T_B}^*,\ {I_A}^* \ge {I_B}^*\ and\ {F_A}^* \ge {F_B}^*$.

3.4 Definition

Consider a nonempty setX, and two starneutrosophicfuzzy sets A^* , and B^*; then the star intersection and star union of any two starneutrosophic sets are defined as follows:

1. The star neutrosophic intersection of A^*, $B^*$is defined as :
   Type 1:$A^* \cap B^* = \langle min({T_A}^*, {T_B}^*), min({I_A}^*, {I_B}^*)\ ,\ max({F_A}^*, {F_B}^*)\rangle$.
   Type 2:$A^* \cap B^* = \langle min({T_A}^*, {T_B}^*), max({I_A}^*, {I_B}^*)\ ,\ max({F_A}^*, {F_B}^*)\rangle$.

2. The star neutrosophic union of A^*, $B^*$is defined as :
   Type 1:$A^* \cup B^* = \langle max({T_A}^*, {T_B}^*), max({I_A}^*, {I_B}^*)\ ,\ min({F_A}^*, {F_B}^*)\rangle$.

   Type 2:$A^* \cup B^* = \langle max({T_A}^*, {T_B}^*), min({I_A}^*, {I_B}^*)\ ,\ min({F_A}^*, {F_B}^*)\rangle$.

DOI: 10.5281/zenodo.3930146

3. The star neutrosophic symmetric difference of A^*, B^* is defined as :

$$A^* \ominus B^* = (A^* - B^*) \cup (B^* - A^*), \text{ or equivalently}$$

$$A^* \ominus B^* = (A^* \cup B^*) - (A^* \cap B^*).$$

3.5 Definition

The star neutrosophic difference of any two star neutrosophic fuzzy sets A^* , and $B^*$is to be defined as follows:

Type1: $A^* - B^* = \langle max(0, A^* - B^*), max(0, A^* - B^*), min(1, 1 - (A^* - B^*)) \rangle$.

Type2: $A^* - B^* = \langle max(0, A^* - B^*), min(1, 1 - (A^* - B^*)), min(1, 1 - (A^* - B^*)) \rangle$.

3.1 Example

Let the neutrosophic fuzzy sets $A = < 0.7, 0.4, 0.5 >$ and $B = < 0.8, 0.6, 0.5 >$ then the star neutrosophic fuzzy sets $A^* =< 0.5, 0.3, 0.3 >$, The star neutrosophic complement of A^*, B^*may be equals two types :Type1 and Type 2 $co\ A^* = \langle 0.5, 0.7, 0.7 \rangle$. or $co\ A^* = \langle 0.3, 0.7, 0.5 \rangle$, $B^* =< 0.4, 0.2, 0.2 >$, $co\ B^* = \langle 0.6, 0.8, 0.8 \rangle$. or $co\ B^* = \langle 0.2, 0.8, 0.4 \rangle$. The star neutrosophic union of A^*, B^*may be equals two types :Type 1:$A^* \cup B^* = \langle 0.5, 0.3, 0.2 \rangle$ and Type 2:$A^* \cup B^* = \langle 0.5, 0.2, 0.2 \rangle$. It is easy to calculate other operations

* **The following figure represents the relationship between Types of Sets**

$$\begin{tikzcd}
\text{Crisp Set} \arrow[r] \arrow[d] & \text{Fuzzy Set} \arrow[d] \\
\text{Intuitionistic Crisp Set} \arrow[r] \arrow[d] & \text{Intuitionistic Fuzzy Set} \arrow[d] \\
\text{Neutrosophic Crisp Set} \arrow[r] \arrow[d] & \text{Neutrosophic Fuzzy Set} \arrow[d] \\
\text{Star Neutrosophic Crisp Set} \arrow[r] \arrow[d] & \text{Star Neutrosophic Fuzzy Set} \arrow[d] \\
\text{Retract Neutrosophic Crisp Set} \arrow[r] & \text{Retract Neutrosophic Fuzzy Set}
\end{tikzcd}$$

Fig. 3.1 The Relationsship between Types of Sets

4 Star Neutrosophic Topological Spaces

In this section we introduce a new type of Neutrosophic Fuzzy Topological Spaces, based on the star neutrosophic fuzzy sets.

5.1 Definition

Let X be a non empty set, and τ^*be a family of retract neutrosophic subsets of X, then τ^* is said to be a star neutrosophic topology if it satisfies the following axioms:

O1: $\emptyset_N^*$, $X_N^* \in \tau^*$

O2: $A^* \cap B^* \in \tau^*$, $\forall A^*, B^* \in \tau^*$

O3:$\cup_{j \in J} A_j^* \in \tau^*$, $\forall A_j^* \in \tau^*$, $j \in J$.

5.2 Remarks

DOI: 10.5281/zenodo.3930146

Remark1: The pair (X, τ^{*}) is called a star neutrosophic topological space in X.
Remark2: The elements of $\tau^{*}$are called star neutrosophic open sets in X.
Remark3: A star neutrosophic crisp set F^{*} is said to be star neutrosophic closed set if and only if its complement, co F^{*}, is a star neutrosophic open set

5.3 Definition

Let$(X, \tau_1^{*})$and (X, τ_2^{*})be two star neutrosophic topological spaces in X. Then τ_1^{*} is contained in τ_2^{*} (symbolize ($\tau_1^{*} \subseteq \tau_2^{*}$) if $G \in \tau_2^{*}$ for each $G \in \tau_1^{*}$. In this case, we say that $\tau_1^{*}$is coarser than τ_2^{*} and that τ_2^{*} is said to be finer than τ_1^{*}.

5.4 Definition

Let (X, τ^{*})be aretract neutrosophic topological space, and A^{*}be a star neutrosophic set in X, then the star neutrosophic interior ($int(A^{*})$) of $A^{*}$and the star neutrosophic closure of A^{*} ($cl(A^{*})$) are defined by:

(a) $int(A^{*}) = \cup\{G^{*}: G^{*}$is star neutrosophic open set in X and $G^{*} \subseteq A^{*}\}$;
(b) $cl(A^{*}) = \cap\{F^{*}: F^{*}$ is star neutrosophic closed set in X and$A^{*} \subseteq F^{*}\}$;

Hence $int(A^{*})$ is a star neutrosophic open set in X, that is $int(A^{*}) \in \tau^{*}$, and $cl(A^{*})$ is a star neutrosophic closed set in X, that is $cl(A^{*}) \in \tau^{c^{*}}$.

5.5 Definition

Let X be a non empty set, and f^{*} be a family of star neutrosophic subsets of X, then f^{*} is said to be a star neutrosophic co- topology if it satisfies the following axioms:

C1: $\emptyset_N^{*}, X_N^{*} \in f^{*}$
C2: $A^{*} \cup B^{*} \in f^{*}, \quad \forall A^{*}, B^{*} \in f^{*}$
C3: $\cap_{j \in J} A_j^{*} \in f^{*}, \quad \forall A_j^{*} \in f^{*},\ j \in J.$

5.6 Proposition

Let (X, τ) be a neutrosophic topology spaces and A^{*}, B^{*}be two neutrosophic sets in X holding the following properties:

a) $A^{*}$is star neutrosophic open if and only if$A^{*} = int(A^{*})$;
b) $int(A^{*}) \subseteq A^{r}$, and $A^{*} \subseteq cl(A^{*})$;
c) $A^{*} \subseteq B^{*} \Rightarrow int(A^{*}) \subseteq int(B^{*})$;
d) $int(A^{*} \cap B^{*}) = int(A^{*}) \cap rint(B^{*})$;
e) $cl(A^{*} \cup B^{*}) = cl(A^{*}) \cup cl(B^{*})$;

* **The following figure represents the relationship between topological structures**

$$
\begin{tikzcd}
\text{Crisp Topology} \arrow[r] \arrow[d] & \text{Fuzzy Topology} \arrow[d] \\
\text{Intuitionistic Crisp Topology} \arrow[r] \arrow[d] & \text{Intuitionistic Fuzzy Topology} \arrow[d] \\
\text{Neutrosophic Crisp Topology} \arrow[r] \arrow[d] & \text{Neutrosophic Fuzzy Topology} \arrow[d] \\
\text{Star Neutrosophic Crisp Topology} \arrow[r] \arrow[d] & \text{Star Neutrosophic Fuzzy Topology} \arrow[d] \\
\text{Retract Neutrosophic Crisp Topology} \arrow[r] & \text{Retract Neutrosophic Fuzzy Topology}
\end{tikzcd}
$$

Fig. 4.1 The Relationssship between Types of Topological Structures

DOI: 10.5281/zenodo.3930146

Conclusion

In this paper, a new generalization of the star intuitionistic fuzzy topological space called the star neutrosophic fuzzy topological space was introduced. we've presented the concepts of star neutrosophic fuzzy topological space, star neutrosophic fuzzy cotopological space, star neutrosophic fuzzy closure, and star neutrosophic fuzzy interior.

References

[1] Atanassov, K. "Intuitionistic fuzzy sets: past, present and future", Proc. of the Third Conf. of the European Society for Fuzzy Logic and Technology EUSFLAT 2003, Zittau,pp.12 – 19. 2003.

[2] Atanassov, K. "Intuitionistic fuzzy sets", Fuzzy Sets and Systems 20, pp.87 – 96,1986.

[3] Cornelis,C, Atanassov, Kerre K, E. E." Intuitionistic fuzzy sets and interval-valued fuzzy sets: A critical comparison", Proc. EUSFLAT03, pp.159 – 163, 2003.

[4] SALAMA, A. A.; EL GHAWALBY, HEWAYDA; NASR, ASMAA M." RETRACT NEUTROSOPHIC CRISP SYSTEM FOR GRAY SCALE IMAGE, Asian Journal of Mathematics and Computer Research, 2, pp.104-117,2018

[5] Salama, A. A, Smarandache, F, Alblowi, S. A." New neutrosophic crisp topological concepts", Neutrosophic Sets and Systems 2, pp.50 – 54, 2014.

[6] Salama, A. A., Alblowi, S. "Neutrosophic set and neutrosophic topological spaces", ISOR J. Mathematicsvol. 3, no. 3, pp.31 – 35, 2012.

[7] Salama, A. A., and Alblowi, S."Intuitionistic Fuzzy Ideals Topological Spaces", Advances in Fuzzy Mathematics , Vol.7(1), pp 51- 60, 2012.

[8] Salama, A. A.,Elghawalby,H. "*- Neutrosophic Crisp Set & relations", Neutrosophic Sets and Systems, Vol. 6, 2014.

[9] Smarandache, F, "Neutrosophy and neutrosophic logic", in: First International Conference on Neutrosophy , Neutrosophic Logic, Set, Probability, and Statistics, University of New Mexico, Gallup, NM 87301,USA.

[10] Smarandache, F. "A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability", American Research Press, 1999.

[11] Smarandache, F." Neutrosophic set, a generialization of the intuituionistics fuzzy sets", Inter. J. Pure Appl. Math., 24,pp.287 – 297, 2005.

[12] Zadeh, L. A., "Fuzzy sets", Inform. and Control, pp.338 – 353, 1965.

[13] Broumi S., Bakali A., Talea M., Smarandache F. and Vladareanu L., Applying Dijkstra Algorithm for Solving Neutrosophic Shortest Path Problem, Proceedings of the 2016 International Conference on Advanced Mechatronic Systems, Melbourne, Australia, pp.412-416, 2016.

[14] Broumi S., Talea M., Bakali A., Smarandache F., Application of Dijkstra algorithm for solving interval valued neutrosophic shortest path problem, 2016 IEEE Symposium Series on Computational Intelligence (SSCI), pp.1 – 6, 2016.

DOI: 10.5281/zenodo.3930146

[15] Broumi S., Bakali A., Mohamed T., Smarandache F. and Vladareanu L., Shortest Path Problem Under Triangular Fuzzy Neutrosophic Information, 10th International Conference on Software, Knowledge, Information Management & Applications (SKIMA), pp.169-174, 2016.

Neutrosophic Quotient Algebra

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Abstract

The algebraic properties of neutrsophic ideals over algebra, isomorphism properties of neutrosophic ideal and neutrosophic modules over algebra are discussed in this paper. Some of the charactrisations of Neutrosophic quotient algebra are derived and the role of algebraic structures is studied in the context of neutrosophic set. This paper expands the definition of quotient algebra within the context of neutrosophical set.

Keywords: Neutrosophic algebra over a neutrosophic subfield, Neutrosophic ideal, Neutrosophic quotient algebra.

1.Introduction

In classical set theory, the membership of elements in a set is assessed in binary terms 0 and 1; according to a bivalent condition-an element either belongs or does not belong to the set. As an extension, fuzzy set theory permits the gradual assessment of the membership of elements in a set. A fuzzy set A in X is characterised by a membership function which is associated with each element in X, a real number in the interval [0,1]. Lotfi A Zadeh [1] introduced a theory whose objects fuzzy sets-are sets with imprecise boundaries which allow us to represent vague concepts and contexts in natural language. Fuzzy set theory is limited to modelling a situation involving uncertainty. As an extension of fuzzy set concept, the theory of intuitionistic fuzzy sets introduced whose elements have degree of membership and non membership. Intuitionistic fuzzy sets have been introduced by Krassimir Atanassov [2] as an extension of Lotfi Zadeh's notion of fuzzy set. Let us have a fixed universe X and A is a subset of X .The intuitionistic fuzzy set can be defined as $A = \{(x, \mu_A(x), \nu_A(x) / x \in X\}$ where $0 \leq \mu_A(x) + \nu_A(x) \leq 1$. μ for membership and ν for non membership, which belongs to the real unit interval [0,1] and sum belongs to the same interval.

Neutrosophy is a new branch of philosophy and logic introduced by Florentin Smarandache [3,4] in 1995 which studies the origin and features of neutralities in nature. Each proposition in Neutrosophic logic is approximated to have the percentage of truth (T), the percentage of indeterminacy (I) and the percentage of falsity (F). So this Neutrsophic logic is called generalization of classical logic, conventional fuzzy logic, intuitionistic fuzzy logic and interval valued fuzzy logic. This mathematical tool is used to handle problems like imprecise, indeterminate and inconsistent data. The use of neutrosophic theory becomes inevitable when a situation involving indeterminacy is to be modelled. The introduction of Neutrosophic theory has led to the establishment of the concept of

DOI: 10.5281/zenodo.3930148
Received: May 07, 2019 Revised: July 25, 2019 Accepted: octobre 02, 2019

neutrosophic algebraic structures in this article [5,6]. For more information on real applications of neutrosophic theory, the readers can see [13-15]

The main objective of the neutrosophic set is to narrow the gap between the vague, ambiguous and imprecise real-world situations. Among the different branches of applied and pure mathematics, abstract algebra was one of the first few area where research was conducted using the concept of neutrosophic set. Initially, B. Vasantha Kandasamy and Florentin Smarandache [7] introduced and applied fundamental algebraic neutrosophic structures. This paper focuses on algebra over a field , quotient algebra over a field and algebraic structures ideal in neutrosophic domain and derive some algebraic properties. This paper focuses on algebra over a field , quotient algebra over a field and algebraic structure ideal in neutrosophic domain and derives some algebraic properties.

2. Preliminaries

Abraham Robinson [8] introduced the non-standard analysis in the 1960s, a formalization of the analysis and a branch of mathematical logic that describes the infinitesimals. Informally, an infinitesimal is an infinitely small number. Formally, x is said to be infinitesimal if and only if for all positive integers n one has $|x| < 1/n$. Let $\varepsilon > 0$ be a infinitesimal number. Let us consider the non-standard finite numbers $1^+ = 1+\varepsilon$, where 1 is its standard part and ε its non-standard part, and $^-0 = 0-\in$, where 0 is its standard part and $\in$ its non-standard part. Then, we call] - 0, 1+ [a non-standard unit interval. Obviously, 0 and 1, and analogously non-standard numbers infinitely small but less than 0 or infinitely small but greater than 1, belong to the non-standard unit interval. Generally the left and right borders of non standard interval $]^-a,b^+[$ are vague, imprecise and themselves being a non standard subsets.

Definition 2.1 [9,10] A Neutrosophic set A on the universal set X is defined as $A = \{\langle x, t_A(x), i_A(x), f_A(x)\rangle\}$ where $x \in X$ and $t_A, i_A, f_A : X \to]0^-,1^+[$ where $t, i \,\&\, f$ are known as Neutrosophic components which are subsets of $]0^-,1^+[$ and $^-0 \le t_A(x) + i_A(x) + f_A(x) \le 3^+$. A Neutrosophic set A can be written as $A = \sum_i^n (t_A(x_i), i(x_i), f(x_i)) / x_i$, $x_i \in X$. Thus a Neutrosophic set has 3 components.

i) t represents membership value (Percentage of truth)

ii) i represents indeterminacy (Percentage of indeterminacy)

iii) f represents non membership value (Percentage of falsity)

Since the membership function t_A, i_A, f_A defined in real life and scientific applications are from X in to the unit interval $[0,1]$ as $t_A, i_A, f_A : X \to [0,1]$, a Neutrosophic set A will be denoted by a mapping defined by $A : X \to [0,1] \times [0,1] \times [0,1]$.

Example 2.1 Assume that $X = \{x, y, z\}$, $x$is hard work, y is capability and z is knowledge in particular area. They are obtained from the questionnaire of some domain experts about the question good researcher. A is single valued Neutrosophic set of X defined by

DOI: 10.5281/zenodo.3930148

$$A = {(0.4,0.5,0.2)}/{x} + {(0.5,02.0.3)}/{y} + {(0.6,0.2,0.3)}/{z}$$

Definition 2.2 [3,9] Let A and B be two Neutrosophic sets on X. Then

i) A is contained in B, denoted as $A \subseteq B$ if and only if $A(x) \leq B(x)$, $\forall x \in X$ this means that

$$t_A(x) \leq t_B(x), i_A(x) \leq i_B(x) \,\&\, f_A(x) \geq f_B(x)$$

ii) The union of A and B is denoted by $C = A \bigcup B$ and defined as $C(x) = A(x) \vee B(x)$ where

for each $x \in X$. This means that $A(x) \vee B(x) = \{t_A(x) \vee t_B(x), i_A(x) \vee i_B(x), f_A(x) \vee f_B(x)\}$. i.e.

$$t_C(x) = \max\{t_A(x), t_B(x)\}, i_C(x) = \max\{i_A(x), i_B(x)\} \,\&\, f_C(x) = \min\{f_A(x), f_B(x)\}$$

iii) The intersection of A and B is denoted by $C = A \bigcap B$ and defined as $C(x) = A(x) \wedge B(x)$

where $A(x) \wedge B(x) = \{t_A(x) \wedge t_B(x), i_A(x) \wedge i_B(x), f_A(x) \wedge f_B(x)\}$ for each $x \in X$.

i.e $t_C(x) = \min\{t_A(x), t_B(x)\}, i_C(x) = \min\{i_A(x), i_B(x)\} \,\&\, f_C(x) = \max\{f_A(x), f_B(x)\}$

iv) The compliment of A is denoted by A^C and defined as $A^C(x) = (f_A(x), 1 - i_A(x), t_A(x))$,

for each $x \in X$. Here $(A^C)^C = A$

Definition 2.3 [11] An algebra is an algebraic structure which consist of a set, together with multiplication, addition and scalar multiplication by elements of underline field and satisfies the axioms implied by vector field and bilinear. An algebra over a field is a vector space equipped with bilinear product.

Definition 2.4 [12] Let V be a vector space over a field F equipped with binary operation from $V \times V \to V$. Then V is an algebra over a field F if the following conditions hold $\forall x, y, z \in V$ and $a, b \in F$

- $(x + y).z = x.z + y.z$
- $z.(x + y) = z.x + z.y$
- $(ax).(by) = (ab)(x.y)$

3 Neutrosophic quotient algebra

This section defines neutrosophic quotient algebra over a field and derive some elementary properties by extending the concept of algbra over a field in neutrosophic set.

Definition 3.1 Let A be algebra over field F, then the neutrosophic subset N_A of A is called neutrosophic algebra over F if for all $x, y \in A, \alpha \in F$, we have

DOI: 10.5281/zenodo.3930148

i) $N_A(x-y) \geq N_A(x) \wedge N_A(y)$

$\Rightarrow t_A(x-y) \geq t_A(x) \wedge t_A(y), i_A(x-y) \geq i_A(x) \wedge i_A(y), f_A(x-y) \leq f_A(x) \vee f_B(y)$

ii) $N_A(xy) \geq N_A(x) \wedge N_A(y)$

$\Rightarrow t_A(xy) \geq t_A(x) \wedge t_A(y), i_A(xy) \geq i_A(x) \wedge i_A(y), f_A(xy) \leq f_A(x) \vee f_B(y)$

iii) $N_A(\alpha x) \geq N_A(x)$

$\Rightarrow t_A(ax) \geq t_A(x), i_A(\alpha x) \geq i_A(x) \wedge i_A(y), f_A(\alpha x) \leq f_A(x)$

iv) $N_A(0) = 1$

$\Rightarrow t_A(0) = 1, i_A(0) = 1, f_A(0) = 0$

Definition 3.2 Let N_F be a neutrosophic subset of a Field F. If $\lambda_1, \lambda_2 \in F$,

i) $N_F(\lambda_1 - \lambda_2) \geq N_F(\lambda_1) \wedge N_F(\lambda_2)$

$\Rightarrow t_F(x-y) \geq t_F(x) \wedge t_F(y), i_F(x-y) \geq i_F(x) \wedge i_F(y), f_F(x-y) \leq f_F(x) \vee f_F(y)$

ii) $N_F(\lambda_1 \lambda_2^{-1}) = N_F(\lambda_1) \wedge N_F(\lambda_2)$

$\Rightarrow t_F(\lambda_1 \lambda_2^{-1}) \geq t_F(\lambda_1) \wedge t_F(\lambda_2), i_F(\lambda_1 \lambda_2^{-1}) \geq i_F(\lambda_1) \wedge i_F(\lambda_2), f_F(\lambda_1 \lambda_2^{-1}) \leq f_F(\lambda_1) \vee f_F(\lambda_2)$

then N_F is called neutrosophic subfield of F

Definition 3.3 Let A be an algebra over a field F and N_F be a neutrosophic subfield of a Field F. A neutrosophic subset N_A of A is called neutrosophic algebra N_F^A of A over the neutrosophic subfield N_F if it satisfies the following condition. If for $a_1, a_2 \in A$ and $\lambda \in F$

i) $N_A(a_1 - a_2) \geq N_A(a_1) \wedge N_A(a_2)$

$\Rightarrow t_A(a_1 - a_2) \geq t_A(a_1) \wedge t_A(a_2), i_A(a_1 - a_2) \geq i_A(a_1) \wedge i_A(a_2), f_A(a_1 - a_2) \leq f_A(a_1) \vee f_A(a_2)$

ii) $N_A(\lambda a_1) \geq N_F(\lambda) \wedge N_A(a_1)$

$\Rightarrow t_A(\lambda a_1) \geq t_F(\lambda) \wedge t_A(a_1), i_A(\lambda a_1) \geq i_F(\lambda) \wedge i_A(a_1), f_A(\lambda a_1) \leq f_F(\lambda) \vee f_A(a_1)$

iii) $N_A(a_1 a_2) \geq N_A(a_1) \wedge N_A(a_2)$

DOI: 10.5281/zenodo.3930148

$\Rightarrow t_A(a_1a_2) \geq t_A(a_1) \wedge t_A(a_2), i_A(a_1a_2) \geq i_A(a_1) \wedge i_A(a_2), f_A(a_1a_2) \leq f_A(a_1) \vee f_A(a_2)$

It is denoted as neutrosophic algebra N_F^A

Definition 3.4 Let U be a neutrosophic algebra N_F^A. If for $a_1, a_2 \in A$ and $\lambda \in F$

i) $U(a_1a_2) \geq U(a_1) \wedge U(a_2)$

$$\Rightarrow t_U(a_1a_2) \geq t_U(a_1) \wedge t_U(a_2), i_U(a_1a_2) \geq i_U(a_1) \wedge i_U(a_2), f_U(a_1a_2) \leq f_U(a_1) \vee f_U(a_2)$$

ii) $U(\lambda a_1) \geq U(\lambda) \wedge U(a_1)$

$$\Rightarrow t_U(\lambda a_1) \geq t_F(\lambda) \wedge t_U(a_1), i_U(\lambda a_1) \geq i_F(\lambda) \wedge i_U(a_1), f_U(\lambda a_1) \leq f_F(\lambda) \vee f_U(a_1)$$

then U is called neutrosophic N_F^A ideal

Definition 3.5 Let A be neutrosophic N_X^Y-ideal and Y be an algebra over a field X, then $y \in Y$, define neutrosophic subset $(Y + A)(y_1) = A(y_1 - y), y_1 \in Y$

Preposition 3.1 Let A be neutrosophic N_X^Y-ideal, then for all $y_1, y_2 \in Y$,
$y_1 + A = y_2 + A \Leftrightarrow A(y_1 - y_2) = A(0)$

Proof

Necessary part

Given $y_1 + A = y_2 + A$

$(y_2 + A)(y_1) = A(y_1 - y_2) \ldots (1)$

$(y_1 + A)(y_1) = A(y_1 - y_1) = A(0) \ldots (2)$

i.e., $y_1 + A = y_2 + A \rightarrow A(y_1 - y_2) = A(0)$

Sufficient part

Consider $(y_1 + A)(y) = A(y - y_1)$

$(y_1 + A)(y) = A((y - y_2) - (y_1 - y_2))$

$(y_1 + A)(y) \geq A((y - y_2) \wedge A(y_1 - y_2))$

$(y_1 + A)(y) \geq A((y - y_2) \wedge A(0) = (y_2 + A)(y)$

$y_1 + A \geq y_2 + A \ldots (3)$

DOI: 10.5281/zenodo.3930148

Similarly we can prove that $y_2 + A \geq y_1 + A$ ... (4)

From (3) and (4) $y_1 + A = y_2 + A \Leftrightarrow A(y_1 - y_2) = A(0)$

Proposition 3.2: Let A be neutrosophic N_X^Y-ideal, then for all $y_1, y_2, x_1, x_2 \in Y, \lambda \in X$ then

i). $x_1 + A = y_1 + A, x_2 + A = y_2 + A \Rightarrow (x_1 + x_2) + A = (y_1 + y_2) + A$

ii). $x_1 x_2 + A = y_1 y_2 + A$

iii). $x_1 + A = y_1 + A \Rightarrow \lambda x_1 + A = \lambda y_1 + A$

Proof. i) and ii)

$$A((x_1 + x_2) - (y_1 - y)) \geq A(x_1 - y_1) \wedge A(x_2 - y_2) = A(0)$$
$$A(x_1 x_2 - y_1 y_2) = A((x_1 - y_1)x_2 + y(x_2 - y_2)) \geq A(x_1 - y_1) \wedge A(x_2 - y_2) = A(0)$$

So, $A((x_1 + x_2) - (y_1 + y_2)) = A(x_1 x_2 - y_1 y_2) = A(0)$

From preposition 1, $(x_1 + x_2) + A = (y_1 + y_2) + A$ and $x_1 x_2 + A = y_1 y_2 + A$

iii). $x_1 + A = y_1 + A \Rightarrow \lambda x_1 + A = \lambda y_1 + A$

Proof

$$A(\lambda x_1 - \lambda y_1) = A(\lambda(x_1 - y_1)) \geq A(x_1 - y_1) = A(0)$$

i.e. $A(\lambda x_1 - \lambda y_1) = A(0)$, hence $\lambda x_1 + A = \lambda y_1 + A$

Preposition 3.3 Let A be neutrosophic N_X^Y-ideal, then Y/A is algebra over X and $Y/A \cong Y/A_0$ where $Y/A = \{y + A \setminus y \in A\}, A_0 = \{y \in y / A(y) = A(0)\}$ and

$$(y_1 + A) + (y_2 + A) = (y_1 + y_2) + A$$
$$(y_1 + A)(y_2 + A) = y_1 y_2 + A$$
$$\lambda(y_1 + A) = \lambda y_1 + A, \forall y_1, y_2 \in Y, \lambda \in X$$

Proof.

From prepositions stated above ,we can conclude that Y/A is an algebra over X and $f : Y/A \to Y/A_0$ defined by $f(y + A) = y + A_0$ is an isomorphism. Hence $Y/A \cong Y/A_0$

DOI: 10.5281/zenodo.3930148

Definition 3.3 Let A be a neutrosophic N_X^Y ideal, then Y/A is called neutrosophic quotient algebra of Y concern with A

5. Conclusions

Neutrosophic quotient algebra is one of the generalizations of quotient algebra. This paper has developed a combination of an algebraic structure , quotient algebra with neutrosophic set theory. Neutrosophic quotient algebra becomes a key element in the study of neutrosophic quotient modules of an R- module and their properties. This study leads to algebraic nature of neutrosophic algebraic structure and the evolution of new neutrosophic algebraic structures.

Funding: "This research received no external funding"

Conflicts of Interest: "The authors declare no conflict of interest."

References

[1] Zadeh, Lotfi A. "Fuzzy sets." *Information and control* 8.3, 338-353, 1965.
[2] Atanassov, Krassimir. "Intuitionistic fuzzy sets." *International Journal Bioautomation* 20, 2016.
[3] F. Smarandache. *Neutrosophy / Neutrosophic probability, set, and logic*, American Research Press, 1998. See also: http://gallup.unm.edu/~smarandache/NeutLog.txt.
[4] Smarandache, Florentin. "Neutrosophic set-a generalization of the intuitionistic fuzzy set." *International journal of pure and applied mathematics* 24.3, 287, 2005.
[5] Smarandache, Florentin, ed. *A unifying field in logics: Neutrosophic logic. neutrosophy, neutrosophic set, neutrosophic probability: Neutrosophic logic: neutrosophy, neutrosophic set, neutrosophic probability*. Infinite Study, 2003.
[6] Wang, Haibin, et al. *Single valued neutrosophic sets*. Infinite study, 2010.
[7] Kandasamy, WB Vasantha, and Florentin Smarandache. *Basic neutrosophic algebraic structures and their application to fuzzy and neutrosophic models*. Vol. 4. Infinite Study, 2004.
[8] Robinson, Abraham. *Non-standard analysis*. Princeton University Press, 2016.
[9] Smarandache, Florentin, ed. *A unifying field in logics: Neutrosophic logic. neutrosophy, neutrosophic set, neutrosophic probability: Neutrosophic logic: neutrosophy, neutrosophic set, neutrosophic probability*. Infinite Study, 2003.
[10] Smarandache, Florentin. "Neutrosophic set–a generalization of the intuitionistic fuzzy set." *Journal of Defense Resources Management (JoDRM)* 1.1,pp.107-116, 2010.
[11] Hazewinkel, Michiel; Gubareni, Nadiya; Kirichenko, Vladimir V. (2004). Algebras, rings and modules. 1. Springer. ISBN 1-4020-2690-0.
[12] Kunz, Ernst. Introduction to commutative algebra and algebraic geometry. Springer Science & Business Media, 2012.
[13] Broumi S., Son L.H., Bakali A., Talea M., Smarandache F., Selvachandran G., "Computing Operational Matrices in Neutrosophic Environments: A Matlab Toolbox", Neutrosophic Sets and Systems, Vol. 18, pp.58-66, 2017.
[14] Broumi S., Bakali A., Talea M,, and Smarandache F,"Isolated Single Valued Neutrosophic Graphs", Neutrosophic Sets and Systems, Vol. 11, pp.74-78, 2016.
[15] Broumi S., Dey A., Bakali A., Talea M., Smarandache F., Son L. H., Koley D., "Uniform Single Valued Neutrosophic Graphs", Neutrosophic Sets and Systems, Vol. 17, pp.42-49, 2017.

Neutrosophic Crisp β- Functions

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Abstract

The purpose of the present paper is to introduce and study the concept of β-continuous function and β-open function in neutrosophic crisp topological spaces. Finally, some characterizations concerning neutrosophic crisp functions are presented and one obtains several properties.

Keywords: Neutrosophic crisp β-continuous function, Neutrosophic crisp β-open function and Neutrosophic crisp β-closed function

1. Introduction

Smarandache introduced the neutrosophic components T, I, F which represent the membership, indeterminacy, and non-membership values respectively, where $]^-0, 1^+[$ is non-standard unit interval. After the introduction of the neutrosophic crisp set concepts in [1-8] and after haven given the fundamental definitions of neutrosophic crisp set operations. Some applications of neutrosophic theory can be found in [12-16]. We generalize the crisp functions to the notion of neutrosophic crisp set. Finally, we introduce the definitions of neutrosophic crisp β-continuous function and neutrosophic crisp β-open function, and we obtain several properties and some characterizations concerning the neutrosophic crisp topologicl space.

2. Preliminaries

We recollect some relevant basic preliminaries, and in particular, the work of Smarandache in [9-11]. Salama et al. [1-8] considered some possible definitions for basic concepts of the neutrosophic crisp set and its operations. We now improve some results by the following.

Definition 2.1 [4]

DOI: 10.5281/zenodo.3930154

Received: June 03, 2019 Revised: Agust 25, 2019 Accepted: Novembre 01, 2019

For any non-empty fixed set X, a neutrosophic crisp set (NC-set, for short) A is an object having the form A= $\langle A_{1,2},A_3\rangle$, where A_1, A_2 and A_3 are subsets of X satisfying $A_1\cap A_2 =\emptyset$, $A_1\cap A_3 = \emptyset$ and $A_3 \cap A_2 = \emptyset$. Several relations and operations between NC-sets were defined in [3, 6, 8].

Definition 2.2 [3]

A neutrosophic crisp topology (NCT, for short) on a non-empty set X is a family Γ of neutrosophic crisp subsets of X satisfying the following axioms

i) $\emptyset$, $X_N \in \Gamma$.

ii) $A_1 \cap A_2 \in \Gamma$ for any A_1 and $A_2 \in \Gamma$.

iii) $\cup A_j \in \Gamma$ for any $\{A_j : j \in J\}\subseteq \Gamma$.

In this case the pair $(X,)$ is called a neutrosophic crisp topological space ($NCTS$, for short) in X. The elements in Γ are called neutrosophic crisp open sets (NC-open sets for short) in X. A NC-set F is said to be neutrosophic crisp closed set (NC-closed set, for short) if and only if its complement F^c is a NC-open set.

Definition 2.3 [3]

Let $(X,)$ be a $NCTS$ and A=$\langle A_{1,2},A_3\rangle$ be a NC-set in X. Then the neutrosophic crisp closure of A ($NC(A)$ for short) and neutrosophic crisp interior ($NCint(A)$ for short) of A are defined by:

(i) $NCcl(A)$=$\cap\{K$:K is a NC-closed set in X and $A\subseteq K\}$

(ii) $NCint(A)$=$\cup\{G$:G is a NC-open set in X and $G\subseteq A)$,

It can be also shown that $NC(A)$ is a NC-closed set, and $NCint(A)$ is a NC-open set in X .

Definition 2.4 [1]

Let $(X,)$ be a $NCTS$ and A =$\langle A_{1,2},A_3\rangle$ be a NCS in X, then A is called:

i) Neutrosophic crisp α-open set iff $A \subseteq NCi(NCcl(NCint(A))$.

ii) Neutrosophic crisp semi-open set iff $A \subseteq NC(NCint(A))$.

iii) Neutrosophic crisp pre-open set iff $A \subseteq NCi(NCcl(A))$.

iv) Neutrosophic crisp β- open set iff $A \subseteq (NCcl\ (NCi(NCcl\ (A))$.

Definition 2.5 [1]

Let $(X,)$ be a $NCTS$ and A =$\langle A_{1,2},A_3\rangle$ be a NCS in X, and f: $X\longrightarrow X$ then:

1) If $f NC\alpha$-continuous $\Longrightarrow$ inverse image of $NC\alpha$ open set is $NC\alpha$- open set
2) If f NCpre-continuous $\Longrightarrow$ inverse image of NCpre-open set is NCpre- open set
3) If f NCsemi-continuous $\Longrightarrow$ inverse image of NCsemi-open set is NCsemi- open set
4) If f $NC\beta$-continuous $\Longrightarrow$ inverse image of $NC\beta$-open set is $NC\beta$- open set

Definition 2.6 [3]

(a) If A= $\langle A_1,A_2,A_3\rangle$ is a NC-set in X, then the NC-image of A under f denoted by $f(A)$ is the a NC-set in Y defined by $f(A)$= $\langle f(A_1),f(A_2),f(A_3)\rangle$

(b) If f is a bijective map then f:$(X, \Gamma_1)\longrightarrow(Y, \Gamma_2)$ is a map defined such that: for any NC-set B= $\langle B_1,B_2,B_3\rangle$ in Y, the NC-preimage of B, denoted by $f^{-1}(B)$ is a NC-set in X defined by $f^{-1}(B)$= $\langle f^{-1}(B_1),f^{-1}(B_2),f^{-1}(B_3)\rangle$.

Here we introduce the properties of NC-images and NC-preimages, some of which we shall frequently use in the following sections.

Corollary 2.1 [3]

Let A =$\{A_i$:$i\in J\}$ be NC-open sets in X, and B= $\{B_j$: $j\in K\}$ be NC-open sets in Y, and f:$(X, \Gamma_1)\longrightarrow(Y, \Gamma_2)$ a function. Then

(i) $A_1\subseteq A_2 \Leftrightarrow f\ (A_1)\subseteq f\ (A_2)$, $B_1\subseteq B_2 \Leftrightarrow f^{-1}(B_1)\subseteq f^{-1}(B_2)$,

(ii) $A\subseteq f^{-1}((A))$ and if f is injective, then A_1= $f^{-1}((A))$.

(iii) $f^{-1}((B))\subseteq B$ and if f is injective, then $f^{-1}(f(B))$= B.

(iv) $f^{-1}(\cup B_i)$= $\cup f^{-1}(B_i)$, $f^{-1}(\cap B_i)\subseteq \cap f^{-1}(B_i)$,

(v) $f(\cup A_i)$= $\cup f(A_i)$, $f(\cap A_i)\subseteq \cap f(A_i)$.

(vi) $f^{-1}(Y_N)$= X_N, $f^{-1}(\Phi_N)$= Φ_N,

(vii) (Φ_N)= Φ_N, (X_N)= Y_N, if f is subjective.

DOI: 10.5281/zenodo.3930154

3. Neutrosophic crisp β-continuous function

Definition 3.1

A function :(, Γ_1)⟶(Y, Γ_2) is said to be $NC\beta$-continuous (briefly $NC\beta$-cont) if the inverse image of each NC-open set in Y is $NC\beta$-open in X.

It is clear that the class of $NC\beta$-continuity contains each of classes NC-semiopen and NC-preopen the implication between then and other type of continuities are given by the following diagram.

$$\begin{tikzcd}
 & & NC\text{-semi continuity} \arrow[dr] & \\
NC\text{-continuity} \arrow[r] & NC\alpha\text{-continuity} \arrow[ur] \arrow[dr] & & NC\beta\text{-continuity} \\
 & & NC\text{-pre continuity} \arrow[ur] &
\end{tikzcd}$$

The converses of these implication not hold, in general, as shown in the following example.

Example 3.1

Let X=Y= {a,b,c,d} and let the NCT on X be indiscrete and on Y be discrete. The identity function :(, Γ_1)⟶(Y, Γ_2) is $NC\beta$-continuous but not NC-semi continuous.

Example 3.2

Let X=Y={a,b,c,d} with NC-topologies $\Gamma_x=\{X_N,\Phi_N,A\}$, $\Gamma_y=\{Y_N,\Phi_N,D\}$ where

A=⟨{a,b}, {b,d},{c}⟩,

D= ⟨{a,d},{a,d},{c}⟩.

A function :(, Γ_1)⟶(Y, Γ_2) which defined as

(a)=a, (b)=c and f(c)=b f(d)=a, is $NC\beta$-cont. but not NC-pre cont.

The following theorem gives easy characterization of $NC\beta$-continuity.

Theorem 3.3

Each $NC\beta$-open set which is also NCsemi-closed set is NCsemi-open.

Proof. Let A= ⟨A_1,A_2,A_3⟩ be a $NC\beta$-open set which is also $NC\beta$-closed set then, $A\subseteq NCcl\ NCint\ NCclA$ and $A\subseteq NCint\ NCclA$. Thus $NCint\ NCclA\subseteq A\subseteq NC(NCintNCclA)$. Therefore, A= ⟨$A_{1,2}$,A_3⟩ is NC-semiopen.

Theorem 3.4

Let :(,Γ_1)⟶(Y,Γ_2) be a function. The following statement are equivalent.

DOI: 10.5281/zenodo.3930154

(i) f is $NC\beta$-cont.

For each NC-set $x \epsilon X$ and each NC-open set $= \langle V_1, V_2, V_3 \rangle \subseteq Y$ containing $f(x)$, there exists a $NC\beta$-open set $W = \langle W_1, W_2, W_3 \rangle \subseteq X$ containing (x) such that $f(w) \subseteq V$.

(ii) the inverse image of each NC-closed set in Y is $NC\beta$-closed set in X.

(iii) $NCint\ NCcl\ NCi(f^{-1}(A))) \subseteq f^{-1}(NCcl(A))$.

(iv) $f(NCint\ NCcl\ NCint\ D) \subseteq NCcl(f(D))$. For each $D = \langle D_{1,2}, D_3 \rangle \subseteq X$.

Proof. (i)⟹(ii). Since $V = \langle V_{1,2}, V_3 \rangle \subseteq Y$ containing f(x) is NC-open, then $f^{-1}(V) \epsilon NC\beta$(x). NC-set $W = f^{-1}(V)$. which containing X, therefore $f(w) \subseteq V$.

(i)⟸(ii). Let $V = \langle V_{1,2}, V_3 \rangle \subseteq Y$ be NC-open, and let x$\epsilon f^{-1}(V)$, then $f(x) \epsilon V$ and thus there exists $W_x \epsilon NC\beta$(x).such that xϵW_x and $f(W_x) \subseteq V$. then x$\epsilon W_x \subseteq f^{-1}(V)$, and so, $f^{-1}(V) = \cup W_x$, x$\epsilon f^{-1}(V)$. but $\cup W_x \epsilon NC\beta$(x) By Theorem 2.5. hence $f^{-1}(V) \subseteq NC\beta$(x). and therefore f is $NC\beta$-cont.

(i)⟹(iii). Let $F = \langle F_{1,2}, F_3 \rangle \subseteq$ be NC-closed, Y-F is NC-open, $f^{-1}(Y\text{-}F) \epsilon NC\beta$(x). i.e., X-$f^{-1}(F)$ $NC\beta$(x). then $f^{-1}(F) = \langle f^{-1}(F_1), f^{-1}(F_2), f^{-1}(F_3) \rangle$ is $NC\beta$-closed set in X.

(iii) ⟹ (iv). Let $A = \langle A_{1,2}, A_3 \rangle \subseteq Y$, then $f^{-1}(NCcl(A))$ is $NC\beta$-closed set in X, i.e., $f^{-1}(NCcl(A)) \supseteq NCint\ NCcl\ NCint(f^{-1}(NCcl(A))) \supseteq NCint\ NCcl\ NCint(f^{-1}(A)))$.

(iv)⟹(v). Let $D = \langle D_1, D_2, D_3 \rangle \subseteq X$, NC-set $A = f(D)$ in (iv), that $NCintNCcl\ NCint(f^{-1}(f(D)) \subseteq f^{-1}(NCcl(f(D)))$, $NCint\ NCcl\ int(D) \subseteq f^{-1}(NCcl(f(D)))$, then $f(NCint\ NCcl\ NCint(D)) \subseteq (NCcl(f(D)))$.

(v)⟹(i). Let $V = \langle V_1, V_2, V_3 \rangle \subseteq Y$ be a NC-open, $W = Y$-V and $D = f^{-1}(W)$, by $f(NCint\ NCcl\ NCint(f^{-1}(W))) \subseteq NCcl(f(f^{-1}(W))) \subseteq NCcl(W) = W$.

so, $NCint\ NCcl\ NCint(f^{-1}(W)) \subseteq f^{-1}(W)$, i.e., $f^{-1}(W) = \langle f^{-1}(W_1), f^{-1}(W_2), f^{-1}(W_3) \rangle$ is $NC\beta$-closed set in X, thus f is $NC\beta$-cont.

Theorem 3. 5

If $:(, \Gamma_1) \longrightarrow (Y, \Gamma_2)$ be a $NC\beta$-cont. and $NC\alpha$-open function. Then the inverse image of any $NC\beta$-open set in Y is $NC\beta$-open set of X.

Proof. Let $W = \langle W_{1,2}, W_3 \rangle \epsilon NC\beta$(y), then $W \subseteq NCcl\ NCint\ NCcl(W)$ and so, $f^{-1}(W) \subseteq f^{-1}(NCcl\ NCint\ NCcl(W)) \subseteq NCcl(f^{-1}(NCint\ NCcl\ (W)))$. because f is $NC\alpha$-open and $NCint\ NCcl(W)$ is NC-preopen. Since f is $NC\beta$-cont. $f^{-1}(W) \subseteq NCcl\ NCint\ NCcl\ (f^{-1}(NCint\ NCcl\ (W)) \subseteq NCcl\ NCint\ NCcl(f^{-1}(NCcl\ NCint\ NCcl(W)) \subseteq NCcl\ NCint\ NCcl(f^{-1}((W))$ - $NCint\ NCcl(f^{-1}((W))$. Because f is $NC\alpha$-open.

Theorem 3. 6

If $:(, \Gamma_1) \longrightarrow (Y, \Gamma_2)$ be a $NC\beta$-cont. and NC-open function, Then the

following statements hold.

(i) The inverse image of each NC-preopen in Y is $NC\beta$-open in X.

(ii) The inverse image of each NC-semiopen in Y is $NC\beta$-open in X.

DOI: 10.5281/zenodo.3930154

Proof. (i) Let $A=\langle A_1,A_2,A_3\rangle \in NC$-preopen (Y), $A\subseteq NCint\ NCcl\ A$, then $f^{-1}(A)\subseteq NCcl\ NCint\ NCcl(f^{-1}(NCint\ NCclf^{-1}(A)))\subseteq NCcl\ NCint\ NCcl(f^{-1}(NCclf(A)))\subseteq NCcl\ NCint\ NCcl\ f^{-1}(A)= NCcl\ NCint\ NCcl\ f^{-1}(A)$.

(ii) Let $D\in NC$-semiopen (Y), $D\subseteq NCcl\ NCint\ D$, and so, $f^{-1}(D)\subseteq f^{-1}(NCcl\ NCintD\subseteq NCcl(f^{-1}(NCint(D)))\subseteq NCclNCintNCcl(f^{-1}(NCint(D))) = NCclNCintNCclf^{-1}(NCint(D))\subseteq NCcl\ NCint\ NCcl\ f^{-1}(D)$.

Theorem 3. 7

Let $:(, \Gamma_1)\longrightarrow(Y, \Gamma_2)$ be $NC\beta$-cont. surjective such that $NCcl\ NCint\ NCclf^{-1}(V)\subseteq f^{-1}(NCclV)$, for each NC-open set $V=\langle V_1,V_2,V_3\rangle \subseteq$Y. if X is connected, then Y is connected.

Proof. Let Y is not connected, i.e., there exists two NC-open sets $_1$ and V_2 such that $V_1\cup V_2 = Y$ and $V_1\cap V_2 = \phi$. Since f is $NC\beta$-cont. then $f^{-1}(V_i)\subseteq (NCcl\ NCintNCcl(f^{-1}(V_i)))\subseteq f^{-1}(NCcl\ (V_i))= f^{-1}(V_i)$, $i\in \{1,2\}$. so, $\cap f^{-1}(V_i)\subseteq\cap NCclNCintNCcl(f^{-1}(V_i))\subseteq\cap f^{-1}(NCcl(V_i)) \subseteq f^{-1}(\cap V_i) = f^{-1}(\phi)=\phi$, and $\cup f^{-1}(V_i)\subseteq \cup(NCclNCint\ NCcl(f^{-1}(V_i)))\subseteq\cup f^{-1}(NCcl\ V_i))=\cup f^{-1}(V_i)= f^{-1}(\cup V_i)= f^{-1}(Y)= X$.

Therefore X is not connected which leads to a contradiction. Then Y is connected. The relation between $NC\beta$-cont. and θ-cont. will be clear by the following theorem.

Remark 3.1

The composition of two $NC\beta$-cont. functions need not be $NC\beta$-cont. in general, as shown by the following example.

Example 3.3

Let X=Z={a,b,c,d} and Y={a,b,c,d,e}with NC-topologies $\Gamma_x=\{X_N,\Phi_N,A\}$, $\Gamma_y=\{Y_N,\Phi_N,C\}$, $\Gamma_z=\{Z_N,\Phi_N,D\}$ where

A=⟨{a,b},{c},{b}⟩,

C=⟨{d},{c},{a,b}⟩, and

D= ⟨{a,d},{c,d},{a,b}⟩.

Let the identity function $f:(X, \Gamma_1)\longrightarrow(Y, \Gamma_2)$ and $i:(X, \Gamma_2)\longrightarrow(Y, \Gamma_3)$ defined as f(a)=a, f(b)=b=f(d) and f(c)=e. it is clear that each of f and i is $NC\beta$-cont. but $f\circ i$ is not $NC\beta$-cont.

The next theorem shown that under what condition that composition is $NC\beta$-cont.

Theorem 3.8

If $f:(X, \Gamma_1)\longrightarrow(Y, \Gamma_2)$ and g:$(Y, \Gamma_2)\longrightarrow$(Z, Γ_3) be two functions, if f is$NC\beta$-cont. and $NC\alpha$-open and g is a $NC\beta$-cont., then g$o f$ is $NC\beta$-cont.

Proof. Let $V\subseteq$ Z be a NC-open set, then $(gof)^{-1}(V) = f^{-1}(g^{-1}(V)$, but $g^{-1}(V)\in NC\beta$(x) for g is $NC\beta$-cont., and by Theorem 2.6, $f^{-1}(g^{-1}(V)\in NC\beta$(x).

Therefore g$o f$ is $NC\beta$-cont. The following lemma is very useful in the sequel.

Lemma 3.1

If $=\langle U_1,U_2,U_3\rangle \in NC\alpha$-(x) and $V=\langle V_1,V_2,V_3\rangle \in NC\beta$(x), then $U\cap V\in NC\beta(u)$.

Proof. Since $U\cap V\subseteq NCint\ NCcl\ NCintU\cap NCcl\ NCint\ NCclV\subseteq NCcl\ (NCint\ NCcl\ NCintU\cap NCintNC\ clV)\subseteq NCcl\ (NCcl\ NCintU\cap NCint\ NCclV)\subseteq NCcl(NCintU\cap NCint\ NCclV)$, $U\cap V\subseteq NCcl(NCintU\cap\ NCint\ NCclV\cap U = NCcl\ (NCintU\cap NCint\ NCclV)$. but $NCintU\cap\ NCint\ NCclV\subseteq U$ is NC-open in X, then $NCint\ (NCintU\cap NCint$

$NCclV$)= $NCint$ U∩$NCint$ $NCclV$, thus U∩V⊆ $NCcl$ ($NCint$($NCintU$ ∩$NCclV$))⊆ $NCcl$($NCint$ ($NCcl$ ($NCintU$∩V)∩U))⊆ $NCcl$($NCint$($NCcl$ (U∩V)∩U))=$NCcl$($NCint$($NCcl$(U∩V)∩U)).

Therefore U∩V∈ (u).

Theorem 3.9

If :(, Γ_1)⟶(Y, Γ_2) be a $NC\beta$-cont. and $NC\alpha$(x). Then $f\backslash U$ is $NC\beta$-cont.

Proof. Let V = ⟨$V_{1,2}$,V_3⟩⊆ Y be a NC-open set, then $f^{-1}(V)$ ∈ $NC\beta$(x), since U=⟨U_1,U_2,U_3⟩∈ $NC\alpha$(x), by Lemma 2.12 U∩$f^{-1}(V)$ = $(f\backslash U)^{-1}(V)$∈ $NC\beta$(x) therefore $f\backslash U$ is $NC\beta$-cont.

Lemma 3.2

Let A= ⟨$A_{1,2}$,A_3⟩⊆ Y⊆ X, Y∈ $NC\beta$(x) and A∈ $NC\beta$(y), then A∈ $NC\beta$(x).

Proof. Since A= ⟨A_1,A_2,A_3⟩⊆ $NC\beta$(y)⊆ $NCcl$($NCint$($NCcl$(A)))⊆$NCcl$($NCint$($NCcl$(A∪Y)) ⊆ $NCcl$($NCint$($NCcl$(A))). since $NCint$ $NCclA$ is NC-open in Y, then exists a NC-open set U⊆X such that $NCint$ $NCclA$= U∩Y, thus A⊆ $NCcl$(U∩ $NCcl$ $NCint$ $NCclY$)⊆ $NCcl$($NCcl$ $NCint$ $NCcl$(U∩Y)))= $NCcl$($NCcl$ $NCint$ $NCcl$($NCint$ $NCclA$)))⊆ $NCcl$($NCcl$ $NCint$ $NCclA$) ⊆ $NCcl$ $NCint$ $NCclA$= $NCcl$ $NCint$ $NCclA$. Therefore, A⊆ $NC\beta$(x).

Theorem 3.10

If :(, Γ_1)⟶(Y, Γ_2) be a function, and {U_i ,i∈I} be a cover of X by $NC\beta$-open set of X, then f is $NC\beta$-cont. if ($f\backslash U$) is $NC\beta$-cont. for each i∈I.

Proof

Let ⟨$V_{1,2}$,V_3⟩⊆ Y be a NC-open set, then $(f\backslash U)^{-1}(V)$∈ $NC\beta(U_i)$ since U_i∈ $NC\beta$(x). by Lemma 2.12, $(f\backslash U)^{-1}(V)$∈ $NC\beta$(x) for each i∈I. but $f^{-1}(V)$=∪ $(f\backslash U_i)^{-1}$ (V), by Remark 2.9 $f^{-1}(V)$∈ $NC\beta$(x). this implies that f is $NC\beta$-cont.

4. Neutrosophic crisp β-open (closed) function.

Definition 4.1

A function :(, Γ_1)⟶(Y, Γ_2) is said to be $NC\beta$-open If the image of any NC-open set in X is $NC\beta$-open in Y.

Definition 4.2

A function :(, Γ_1)⟶(Y, Γ_2) is said to be $NC\beta$-closed If the image of any NC-closed set in X is $NC\beta$-closed set in Y.

The implications between $NC\beta$-open ($NC\beta$-closed) function and other types of NC-open (NC-closed) function are given by the following diagram.

$$
\begin{tikzcd}
 & & \begin{array}{c} NC\text{-semiopen} \\ (NC\text{-semiclosed}) \end{array} \arrow[dr] & \\
\begin{array}{c} NC\text{-open} \\ (NC\text{-closed}) \\ \text{function} \end{array} \arrow[r] & \begin{array}{c} NC\alpha\text{-open} \\ (NC\alpha\text{-closed}) \\ \text{function} \end{array} \arrow[ur] \arrow[dr] & & \begin{array}{c} NC\beta\text{-open} \\ (NC\beta\text{-closed}) \\ \text{function} \end{array} \\
 & & \begin{array}{c} NC\text{-preopen} \\ (NC\text{-preclosed}) \\ \text{function} \end{array} \arrow[ur] &
\end{tikzcd}
$$

The converses of these statements may be not necessarily true, as shown by the following examples.

Example 4.1

Let X=Y={a,b,c,d} with NC-topologies Γ_x={X_N,Φ_N,A} and Γ_y be an indiscrete NCT.

Where A=⟨{a,b},{c},{b}⟩.

DOI: 10.5281/zenodo.3930154

The identity function f:X⟶Y is $NC\beta$-open ($NC\beta$-closed) but not may be NC-semiopen (NC-semiclosed).

Example 4.2

Let X= {a,b,c,d}, Y= {a,b,c,d,e} with NC-topologies Γ_x={X_N,Φ_N,A}, Γ_y= {Y_N,Φ_N,D}.where A=⟨{a,b},{c},{b,d}⟩, D= ⟨{b,c},{a,c}, {a}⟩.

A function f:X⟶Y defined as (a)= b, f(b)= d and f(c)= e =f(d), it is clear that f is $NC\beta$-open ($NC\beta$-closed) but not NC-preopen (NC-preclosed).

Remark 4.1

A one to one function is $NC\beta$-open iff it is $NC\beta$-closed.

The following theorem gives easy characterization of a $NC\beta$-open function.

Theorem 4.1

Let :(, Γ_1)⟶(Y, Γ_2) be a function. The following statements may be equivalent.

(i) f is $NC\beta$-open.

(ii) For each x∈X and U each neighborhood U of X, there exists a $NC\beta$-open set W= ⟨W_1, $W_{2,3}$⟩⊆ Y containing f(x) such that W⊆ f(u).

Proof. (i)⟹(ii). Let x∈X and U be a neighborhood $U$of X, then there exists a NC-open set = ⟨V_1,V_2,V_3⟩⊆ X such that x∈V⊆U. NC-set W= (V), since f is $NC\beta$-open, (V)=W∈ $NC\beta$(y) and so f(x)∈ W⊆f(u).

(ii)⟹(i). Following directly from the Definition 3.1

Theorem 4.2

Let :(, Γ_1)⟶(Y, Γ_2) be a function. The following statement may be equivalent.

(i) f is $NC\beta$-open.

(ii) $NCint\ NCcl\ NCint\ A$⊆ NC(f^{-1}(A)); for each A⊆ Y.

(iii) if f is bijective, $NCint\ NCcl\ NCint$(f(D))⊆ (f($NCcl$(D)); for each D⊆ X.

Proof. (i)⟹(ii). Since f is $NC\beta$-open and A= ⟨A_1,A_2,A_3⟩⊆ Y, then $NCcl$ (f^{-1}(A))⊆ X containing f^{-1}(A)=⟨f^{-1}(A_1),f^{-1}(A_2),f^{-1}(A_2)⟩ by Theorem 3.9 there is a $NC\beta$-closed set W= ⟨W_1,W_2,W_3⟩ ⊇A such that $NCcl$(f^{-1}(A)) ⊇f^{-1}(W)) ⊇$NCint\ NCcl\ NCint\ f^{-1}$($W$) ⊇$f^{-1}$($NCint\ NCclNCint$ (A)).

(ii)⟹(iii). Let D⊆ X, (D)⊆ Y. NC-set A= f(D) in (ii), then f^{-1}($NCint\ NCcl\ NCint$ (f(D)))⊆ $NCcl$(f^{-1}(f(D))⊆ $NCclD$ and so, $NCint\ NCcl\ NCint$(f(D))⊆ f($NCcl$(D).

(iii)⟹(i). Suppose U=⟨$U_{1,2}$,U_3⟩ is a NC-open set in X, then $NCcl$(f(X-U))=f(X-U) ⊇$NCint\ NCcl\ NCint$(f(X-U)). Since f is bijective, (U) ⊆ $NCint\ NCcl\ NCin$(f(U)) i.e., f(U)∈ $NC\beta$(y), hence f is $NC\beta$-open. Now we try to construct some new connection between $NC\beta$-open ($NC\beta$-closed) functions and other types of NC-open (NC-closed) functions.

Theorem 4.3

Let f:(X, Γ_1)⟶(Y, Γ_2) be a $NC\beta$-open ($NC\beta$-closed) function if W= ⟨W_1, W_2,W_3⟩⊆ Y and F= ⟨F_1,F_2,F_3⟩⊆ X is a NC-open(NC-closed) set containing $f^{-1}(W)$, then there exists $NC\beta$-closed ($NC\beta$-open) H= ⟨H_1,H_2,H_3⟩⊆ Y containing W such that f^{-1}(H)⊆ F.

Proof. H= ⟨H_1,H$_2$,H_3⟩ =Y-f(X-F), since $f^{-1}(W)$⊆ F, W⊆ H, hence H is $NC\beta$-closed and $f^{-1}(H)$ = X-f^{-1}(f(X-F))⊆ F. the second side of the theorem can be prove by the same manner.

Theorem 4.4

If :(, Γ_1)⟶(Y, Γ_2) be a $NC\alpha$-cont and $NC\beta$-open function, Then the inverse image of any $NC\alpha$-open set in Y is $NC\alpha$-open set of X.

Proof. Let V be a $NC\alpha$-open set of Y. so, A∈ $NC\beta$(x), A⊆ $NCint$ $NCcl$ $NCint(A)$ and so, $f^{-1}(V)$⊆ f^{-1}($NCint$ $NCcl$ $NCint(V)$)⊆ $NCint$ $NCcl$ $NCint$(f^{-1}($NCint$ $NCcl$ $NCint(V)$)). Since f is $NC\beta$-open by Theorem 3.7.(ii) we have $f^{-1}(V)$⊆ $NCint$ $NCcl$ $NCint$(f^{-1}($NCint$ $NCcl$ $NCint$ (V)))⊆ $NCint$ $NCcl$ $NCint$($NCcl$(f^{-1}($NCint(V)$)))⊆ $NCint$ $NCclf^{-1}$($NCint(V)$). Since f is NC-αcont., $f^{-1}(V)$⊆ $NCint$ $NCcl$(f^{-1}($NCint$ (V)))⊆ $NCint$ $NCcl$($NCint$ $NCcl$ $NCintf^{-1}$($NCint$ (V)))⊆ $NCint$ $NCcl$ $NCint$($f^{-1}(V)$). Hence $f^{-1}(V)$ is a $NC\alpha$-open set of X.

Corollary 4.1

If :(, Γ_1)⟶(Y, Γ_2) be a $NC\alpha$-cont. and $NC\beta$-open function, Then the inverse image of any $NC\alpha$-closed set in Y is $NC\alpha$-closed set of X.

Proof. Obvious.

Theorem 4.5

If :(, Γ_1)⟶(Y, Γ_2) be a $NC\alpha$-cont. and $NC\beta$-open function, Then the image of any $NC\beta$-open set in X may be $NC\beta$-open set of Y.

Proof. Let A∈ $NC\beta$(x), A⊆ $NCint$ $NCcl$ $NCint(A)$ and so, $f(A)$⊆ f($NCcl$ $NCint$ $NCcl(A)$)⊆ $NCcl$(f($NCint$ $NCcl(A)$))⊆ $NCcl$ $NCint$ $NCcl$ (f($NCint$ $NCcl(A)$))⊆ $NCcl$ $NCint$ $NCcl$ (f($NCcl(A)$))⊆ $NCcl$ $NCint$ $NCclf(A)$= $NCcl$ $NCint$ $NCclf(A)$.

Corollary 4.2

If :(, Γ_1)⟶(Y, Γ_2) be a $NC\alpha$-cont. and $NC\beta$-open and injective, Then the image of each $NC\beta$-closed set in X may be it is $NC\beta$-closed set of Y.

Proof. Let D⊆ X be $NC\beta$-closed, then (X-D) ∊ $NC\beta$(x) by Theorem 3.8 (X-D)⊆ $NCcl$ $NCint$ $NCcl$ (f(X-D)), Y-$f(D)$⊆ Y- $NCcl$ $NCint$ $NCcl$($f(D)$). So, (D) ⊇ $NCint$ $NCclNCin$($f(D)$).

5. Conclusion

In this paper, we introduce both the neutrosophic crisp nearly continuous functions, the neutrosophic crisp nearly open functions, and we present properties related to them. This paper, will promote the future study on

neutrosophic crisp topological functions and many other general frameworks.

6. References

[1] I.M.Hanafy, A.A.Salama, Hewayda ElGhawalby and M.S.Dabash. "Some GIS Topological Concepts via Neutrosophic Crisp Set Theory", To be published in the book titled "New Trends in Neutrosophic Theories and Applications", Publisher: Europa Nova, Brussels, 2016.

[2] A.A.Salama, F. Smarandache and Valeri Kroumov, "Neutrosophic crisp Sets & Neutrosophic crisp Topological Spaces, "Neutrosophic Sets and Systems", Vol.(2), pp.25-30, 2014.

[3] A.A.Salama and Florentin Smarandache, "Neutrosophic crisp set theory", Educational Publisher, Columbus,2015.

[4] A.A. Salama and S.A. Alblowi, "Generalized Neutrosophic Set and Generalized Neutrousophic Topological Spaces ", Journal computer Sci. Engineering, Vol. 2, No. 7, pp. 29-32, 2012.

[5] A.A. Salama and S.A. Alblowi, "Neutrosophic set and neutrosophic topological space", ISORJ. Mathematics, Vol. 3, Issue 4, pp.31-35, 2012.

[6] A. A. Salama,"Neutrosophic Crisp Points & Neutrosophic Crisp Ideals", Neutrosophic Sets and Systems, Vol.1, pp. 50-54. 2013.

[7] A. A. Salama, F. Smarandache, "Filters via Neutrosophic Crisp Sets", Neutrosophic Sets and Systems, Vol.1, pp.34-38. 2013.

[8] A. A. Salama, Said Broumi, Florentin Smarandache,"Some Types of Neutrosophic Crisp Sets and Neutrosophic Crisp Relations" Neutrosophic Theory and Its Applications. 3.2: 396.pp.378-385.

[9] F. Smarandache , "Neutrosophy and Neutrosophic Logic", First International Conference on Neutrosophy , Neutrosophic Logic , Set, Probability, and Statistics University of New Mexico, Gallup, NM 87301, USA 2002 .

[10] F. Smarandache,"An introduction to the Neutrosophy probability applied in Quantum Physics" , International Conference on Neutrosophic Physics , Neutrosophic Logic, Set, Probability, and Statistics University of New Mexico, Gallup, NM 87301, USA, 2-4 December. 2011 .

[11] F. Smarandache, "A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability", American Research Press, Rehoboth, NM, 1999.

[12] S. Broumi, M.Talea, A. Bakali, F. Smarandache, D.Nagarajan, M. Lathamaheswari and M.Parimala, "Shortest path problem in fuzzy, intuitionistic fuzzy and neutrosophic environment: an overview", Complex & Intelligent Systems, 5, pp.371–378, 2019. https://doi.org/10.1007/s40747-019-0098-z

[13] S.Broumi,D. Nagarajan, A. Bakali, M. Talea, F. Smarandache, M. Lathamaheswari, "The shortest path problem in interval valued trapezoidal and triangular neutrosophic environment", Complex & Intelligent Systems, 5, pp.391–402, 2019. https://doi.org/10.1007/s40747-019-0092-5

[14] S. Broumi, D. Nagarajan, A. Bakali, M. Talea, F. Smarandache, M. Lathamaheswari and J. Kavikumar, "Implementation of Neutrosophic Function Memberships Using MATLAB Program", Neutrosophic Sets and Systems, Vol. 27, 44-52, 2019. DOI: 10.5281/zenodo.3275355

[15] S. Broumi, A. Bakali, M. Talea, F. Smarandache, K. P. Krishnan Kishore, R. Şahin, "Shortest Path Problem under Interval Valued Neutrosophic Setting", International Journal of Advanced Trends in Computer Science and Engineering, Vol. 8(1.1), 216-222, 2019.

[16] M. Parimala, M. Karthika, S. Jafari, F. Smarandache, R.Udhayakumar," Neutrosophic Nano ideal topological structure", Neutrosophic Sets and Systems, vol. 24, pp. 70-76, 2019.

DOI: 10.5281/zenodo.3930154